Nadir Jeevanjee

An Introduction to Tensors and Group Theory for Physicists

物理学家用的张量和群论导论

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To My Parents

Preface

This book is composed of two parts: Part I (Chaps. 1 through 3) is an introduction to tensors and their physical applications, and Part II (Chaps. 4 through 6) introduces group theory and intertwines it with the earlier material. Both parts are written at the advanced-undergraduate/beginning-graduate level, although in the course of Part II the sophistication level rises somewhat. Though the two parts differ somewhat in flavor, I have aimed in both to fill a (perceived) gap in the literature by connecting the component formalisms prevalent in physics calculations to the abstract but more conceptual formulations found in the math literature. My firm belief is that we need to see tensors and groups in coordinates to get a sense of how they work, but also need an abstract formulation to understand their essential nature and organize our thinking about them.

My original motivation for the book was to demystify tensors and provide a unified framework for understanding them in all the different contexts in which they arise in physics. The word tensor is ubiquitous in physics (stress tensor, moment-of-inertia tensor, field tensor, metric tensor, tensor product, etc.) and yet tensors are rarely defined carefully, and the definition usually has to do with transformation properties, making it difficult to get a feel for what these objects *are*. Furthermore, physics texts at the beginning graduate level usually only deal with tensors in their component form, so students wonder what the difference is between a second rank tensor and a matrix, and why new, enigmatic terminology is introduced for something they have already seen. All of this produces a lingering unease, which I believe can be alleviated by formulating tensors in a more abstract but conceptually much clearer way. This coordinate-free formulation is standard in the mathematical literature on differential geometry and in physics texts on General Relativity, but as far as I can tell is not accessible to undergraduates or beginning graduate students in physics who just want to learn what a tensor is *without* dealing with the full machinery of tensor analysis on manifolds.

The irony of this situation is that a proper understanding of tensors does not require much more mathematics than what you likely encountered as an undergraduate. In Chap. 2 I introduce this additional mathematics, which is just an extension of the linear algebra you probably saw in your lower-division coursework. This material sets the stage for tensors, and hopefully also illuminates some of the more enigmatic objects from quantum mechanics and relativity, such as bras and kets, covariant and contravariant components of vectors, and spherical harmonics. After laying the necessary linear algebraic foundations, we give in Chap. 3 the modern (component-free) definition of tensors, all the while keeping contact with the coordinate and matrix representations of tensors and their transformation laws. Applications in classical and quantum physics follow.

In Part II of the book I introduce group theory and its physical applications, which is a beautiful subject in its own right and also a nice application of the material in Part I. There are many good books on the market for group theory and physics (see the references), so rather than be exhaustive I have just attempted to present those aspects of the subject most essential for upper-division and graduate-level physics courses. In Chap. 4 I introduce abstract groups, but quickly illustrate that concept with myriad examples from physics. After all, there would be little point in making such an abstract definition if it did not subsume many cases of interest! We then introduce Lie groups and their associated Lie algebras, making precise the nature of the symmetry 'generators' that are so central in quantum mechanics. Much time is also spent on the groups of rotations and Lorentz transformations, since these are so ubiquitous in physics.

In Chap. 5 I introduce representation theory, which is a mathematical formalization of what we mean by the 'transformation properties' of an object. This subject sews together the material from Chaps. 3 and 4, and is one of the most important applications of tensors, at least for physicists. Chapter 6 then applies and extends the results of Chap. 5 to a few specific topics: the perennially mysterious 'spherical' tensors, the Wigner–Eckart theorem, and Dirac bilinears. The presentation of these later topics is admittedly somewhat abstract, but I believe that the mathematically precise treatment yields insights and connections not usually found in the usual physicist's treatment of the subjects.

This text aims (perhaps naively!) to be simultaneously intuitive and rigorous. Thus, although much of the language (especially in the examples) is informal, almost all the definitions given are precise and are the same as one would find in a pure math text. This may put you off if you feel less mathematically inclined; I hope, however, that you will work through your discomfort and develop the necessary mathematical sophistication, as the results will be well worth it. Furthermore, if you can work your way through the text (or at least most of Chap. 5), you will be well prepared to tackle graduate math texts in related areas.

As for prerequisites, it is assumed that you have been through the usual undergraduate physics curriculum, including a "mathematical methods for physicists" course (with at least a cursory treatment of vectors and matrices), as well as the standard upper-division courses in classical mechanics, quantum mechanics, and relativity. Any undergraduate versed in those topics, as well as any graduate student in physics, should be able to read this text. To undergraduates who are eager to learn about tensors but have not yet completed the standard curriculum, I apologize; many of the examples and practically all of the motivation for the text come from those courses, and to assume no knowledge of those topics would preclude discussion of the many applications that motivated me to write this book in the first place.

However, if you are motivated and willing to consult the references, you could certainly work through this text, and would no doubt be in excellent shape for those upper-division courses once you take them.

Exercises and problems are included in the text, with exercises occurring within the chapters and problems occurring at the end of each chapter. The exercises in particular should be done as they arise, or at least carefully considered, as they often flesh out the text and provide essential practice in using the definitions. Very few of the exercises are computationally intensive, and many of them can be done in a few lines. They are designed primarily to test your conceptual understanding and help you internalize the subject. Please do not ignore them!

Besides the aforementioned prerequisites I have also indulged in the use of some very basic mathematical shorthand for brevity's sake; a guide is below. Also, be aware that for simplicity's sake I have set all physical constants such as c and $\hbar$ equal to 1. Enjoy!

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Notation

Some Mathematical Shorthand

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of positive and negative integers
$\in$	"is an element of", "an element of", i.e. $2 \in \mathbb{R}$ reads "2 is an element of the real numbers"
$\notin$	"is not an element of"
$\forall$	"for all"
$\subset$	"is a subset of", "a subset of"
$\equiv$	Denotes a definition
$f : A \to B$	Denotes a map f that takes elements of the set A into elements of the set B
$f : a \mapsto b$	Indicates that the map f sends the element a to the element b
$\circ$	Denotes a composition of maps, i.e. if $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ is given by $(g \circ f)(a) \equiv g(f(a))$
$A \times B$	The set $\{(a, b)\}$ of all ordered pairs where $a \in A$, $b \in B$. Referred to as the *cartesian product* of sets A and B. Extends in the obvious way to n-fold products $A_1 \times \cdots \times A_n$
$\mathbb{R}^n$	$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{ times}}$
$\mathbb{C}^n$	$\underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n\text{ times}}$
$\{A \mid Q\}$	Denotes a set A subject to condition Q. For instance, the set of all even integers can be written as $\{x \in \mathbb{R} \mid x/2 \in \mathbb{Z}\}$
$\square$	Denotes the end of a proof or example

Dirac Dictionary[1]

Standard Notation	Dirac Notation
Vector $\psi \in \mathcal{H}$	$\vert\psi\rangle$
Dual Vector $L(\psi)$	$\langle\psi\vert$
Inner Product (ψ, ϕ)	$\langle\psi\vert\phi\rangle$
$A(\psi)$, $A \in \mathcal{L}(\mathcal{H})$	$A\vert\psi\rangle$
$(\psi, A\phi)$	$\langle\psi\vert A\vert\phi\rangle$
$T_i{}^j e^i \otimes e_j$	$\sum_{i,j} T_{ij}\vert j\rangle\langle i\vert$
$e_i \otimes e_j$	$\vert i\rangle\vert j\rangle$ or $\vert i, j\rangle$

[1] We summarize here all of the translations given in the text between quantum-mechanical Dirac notation and standard mathematical notation.

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