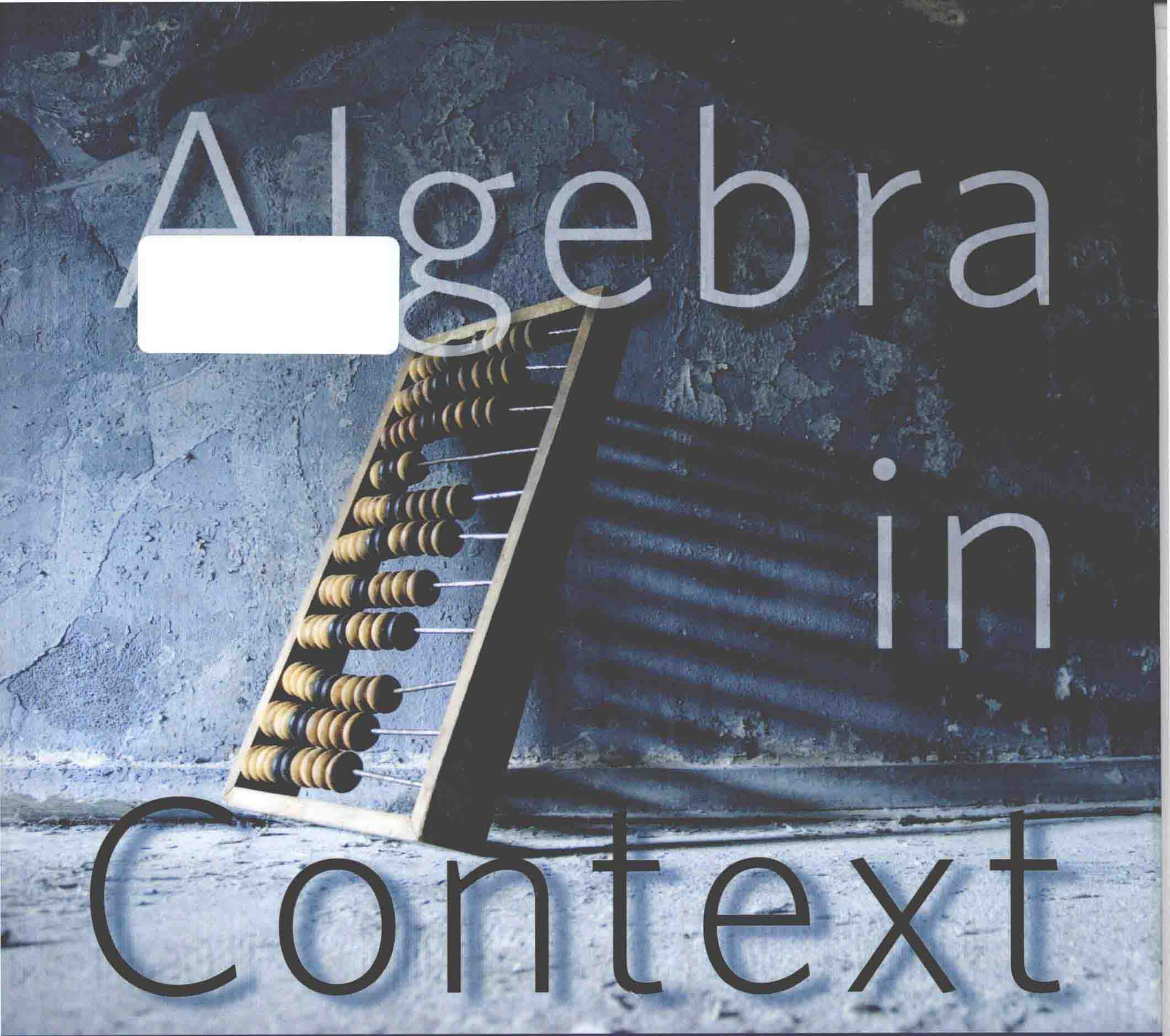
Algebra in Context



Introductory Algebra from Origins to Applications

Amy Shell-Gellasch & J. B. Thoo

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Preface

The history of mathematics is a rich and vibrant area of study that has been drawing increased interest in the mathematical and educational communities over the past few decades. There are two main areas of focus in the history of mathematics: historical research and the uses of the history of mathematics in teaching. Publications in both of these areas have grown every year. An outgrowth of these activities is that there are now many books available on the history of mathematics, ranging from popular books for the general public to textbooks for history of mathematics courses.

There are many ways to partition the books on the history of mathematics. One way is into those that assume that the reader has a calculus background, and those that do not. Textbooks designed primarily for history of mathematics courses (which are required for mathematics and mathematics education majors in many states) generally assume that the reader has a calculus background, and popular books, such as William Dunham's *Journey through Genius* [47] and Eli Maor's *e: The Story of a Number* [80], generally do not. Nevertheless, even popular books on the history of mathematics typically assume that the reader has some mathematics background, usually up through high school algebra. As with history of mathematics textbooks, many resource books also exist to aid college teachers in incorporating the history of mathematics into the mathematics classroom, but to date there is no textbook or resource book that is designed to use history as the vehicle for presenting the mathematical content. Our aim is to provide such a textbook.

You will find many of the topics that are covered in algebra (plus a few other topics such as logic and set theory, including infinite sets); however, unlike a traditional algebra textbook, the topics are presented here in their historical and cultural settings. A fair amount of history that does not pertain directly to mathematics is also presented to give a backdrop to the history of mathematics that is presented. Consequently, this book may be used as a textbook for a course in the history of mathematics that does not assume that the students have a calculus background. It also provides a gateway to appreciating many of the popular books on the history of mathematics if one has never had or is rusty in high school algebra.

This book is suitable for a variety of mathematics courses.

- *General education.* As a textbook for any course that is designed for students who are not science, technology, engineering, or mathematics majors (non-STEM students) but who need to fulfill a quantitative reasoning GE requirement or a multicultural GE requirement. For example, it would work very well for a college algebra for non-STEM majors course or for a liberal arts mathematics course.

- *Mathematics education.* As a textbook for a history of mathematics course that does not have a calculus prerequisite. Many states now require mathematics education majors to take a course in the history of mathematics. Such a course would be especially suitable for future elementary and middle school teachers. Moreover, education majors at any level, including the secondary level, will find the material covered to be useful in their understanding of the underpinnings and the development of mathematics.

- *Any mathematics course.* As a supplementary text for any mathematics course to inject doses of the history of mathematics to bring the course to life. Indeed, after teaching the history of mathematics for many years, both as stand-alone courses and embedded in mathematics courses, we have found that the history of mathematics is a great motivator that encourages students to become engaged in the mathematical topic and to see its uses and beauty. Furthermore, although this book was not designed for use by students who have a calculus background or beyond, more advanced students who have used preliminary versions of it have found that they enjoyed learning the back story, so to speak, of how and why the mathematics they have learned was developed.

- *High school.* The material is also appropriate and accessible to high school students.

ABOUT THIS BOOK

This book is organized into four parts:

Part I	Numeration Systems
Part II	Arithmetic Snapshots
Part III	Foundations
Part IV	Solving Equations

with each part broken into several chapters. The chapters are then broken into sections and possibly subsections. The chapters, sections, and subsections are numbered using Indo-Arabic numerals (1, 2, 3, and so on). For example,

9.2 Grating or Lattice Method	is SECTION 2 of CHAPTER 9
22.3.2 Descartes's Rule of Signs	is SUBSECTION 2 of CHAPTER 22, section 3

Figures and tables are numbered with both the chapter number and the figure or table number. For example,

Figure 2.4: Plimpton 322	is FIGURE 4 in CHAPTER 2
Table 2.1: Babylonian number characters	is TABLE 1 in CHAPTER 2

Throughout the book, we provide exercises in a "just in time" manner to reinforce the material presented. The chapters are sprinkled with two types of exercises: *Now You Try* exercises and *Think About It* exercises. The *Now You Try* exercises give you an opportunity to become familiar with the mathematics that was just discussed, and the *Think About It* exercises ask questions that may require you to ponder. These exercises are integral to your appreciating and understanding the mathematical concepts or the history of mathematics that is presented. There are also additional exercises that are collected in a chapter at the end of each part. Some of these exercises are routine, some are nonstandard to add depth and variety, and some (marked with an $*$) may offer a little more challenge or require you to do a little research. The exercises that require a little research can be used as quick Internet research projects or as ideas for larger projects; they are also designed to be used to motivate class discussions.

The book uses the symbol □ to mark the end of certain blocks. For example,

Remark 1.1 The numbers from one to nine hundred ninety-nine may be considered *fractions of 1000* so that 1000 may be considered the *unit* or basic quantity. For example, eight hundred eleven is $\frac{811}{1000} \times 1000$. As another example, 231,811 is $231\frac{811}{1000} \times 1000$. □

The following table shows the different blocks that end with the □ symbol.

Remark	Think About It	Now You Try	Rule
Example	Definition	Theorem	Corollary

All of these blocks are numbered sequentially within themselves beginning with the chapter number.

To appreciate and to understand any history of mathematics beyond knowing some biographies and anecdotes and sequences of events require understanding some mathematics. This book presents snapshots of the history of mathematics that do not extend beyond high school algebra, from the early beginnings to the eighteenth century. The book assumes that you are already familiar with elementary algebra, specifically, that you are familiar with

- arithmetic with signed numbers (positive and negative numbers)
- the order of operations
- simplifying algebraic expressions (for example, using the distributive law and combining like terms)
- evaluating algebraic expressions when given values of the variables
- solving linear equations in one variable
- solving 2×2 systems of linear equations
- graphing equations in two variables, particularly, graphing linear equations in two variables

that are commonly taught in a traditional high school Algebra I course or in a college remedial algebra course. If you need a refresher or an introduction to these topics, search for "elementary algebra" on the World Wide Web (the Web) to find a slew of materials on the subject. There are also many good videos on the topics on the Web, for example, at `<http://www.mathtv.com>`.

OTHER SOURCES FOR THE HISTORY OF MATHEMATICS

The bibliography lists many references that you may pursue. The references range from covering very specific topics, to specific time periods or geographical regions, to specific or broad themes, to broader surveys and commentaries. Listed among the references are some of the standard textbooks in the history of mathematics, namely, [15, 21, 22, 35, 41, 73, 74, 75]. Although these textbooks assume that the reader has a calculus background, you may still glean a lot from them even if you have to skip over some of the mathematics.

For a very good overview of the history of mathematics that does not assume that the reader has a calculus background, we recommend the introductory chapter, "The History of Mathematics in a Large Nutshell," of Berlinghoff and Gouvêa's excellent book, *Math through the Ages* [11]. Their survey complements the material in this book very nicely.

ACKNOWLEDGMENTS

This book could not have been written and produced without the help of many people.

We thank our colleagues, students, and family for their help, support, and patience while we created what we hope will be a new type of book, and a new approach to teaching basic mathematics.

For reading drafts, early and late, and providing constructive feedback, we thank Erick Gremlich, Christopher Goff, and Travis Smith, and also the anonymous reviewers from Johns Hopkins University Press (JHUP). We also thank the members of the SIGMAA-HOM listserv (the Special Interest Group of the Mathematical Association of America on the History of Mathematics), who very patiently helped clear up some tricky history.

We thank Vincent J. Burke, Catherine Goldstead, and Kathryn Marguy at JHUP for their guidance and support during the production process. Andre M. Barnett did a terrific job of copy-editing. We are, of course, solely responsible for any shortcomings or mistakes.

This book was typeset using the LaTeX memoir documentclass, and that would not have been possible without the encouragement and help from the members of the MacTeX users group. We especially thank William Adams, Nestor E. Aguilera, John Burt, David Derbes, Paul Dulaney, Murray Eisenberg, Gary L. Gray, Martin Wilhelm Leidig, Themis Matsoukas, Scot Mcphee, M. Tamer Özsu, and Axel E. Retif. (We apologize if we left off anyone.) The font family used for the text is TeX Gyre Termes, with the mathematics fonts provided by the `qtxmath` package.

Algebra in Context

Introduction

> An interest in history marks us for life. How we see ourselves and others is shaped by the history we absorb, not only in the classroom but from films, newspapers, television programmes, novels and even strip cartoons. From the time we first become aware of the past, it can fire our imagination and excite our curiosity: we ask questions and then seek answers from history. As our knowledge develops, differences in historical perspectives emerge. And, to the extent that different views of the past affect our perception of ourselves and of the outside world, history becomes an important point of reference in understanding the clash of cultures and of ideas. Not surprisingly, rulers throughout history have recognized that to control the past is to master the present and thereby consolidate their power.
>
> —George Gheverghese Joseph, *The Crest of the Peacock* [72]

> Philosophy is written in this grand book—I mean the Universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it.
>
> —Galileo Galilei, *Assayer* [20]

WHY STUDY THE HISTORY OF MATHEMATICS

These two quotations highlight the complementary aspects of this book. As Joseph notes, "An interest in history marks us for life," for "it can fire our imagination and excite our curiosity: we ask questions and then seek answers from history." Mathematics, as Galileo tells us, is a language. Indeed, mathematics has symbols; it has parts that are equivalent to nouns, verbs, and prepositions; it relies on context and connotation to be understood; it borrows from different cultures and has evolved over time; it even follows fads. The history of mathematics, then, is the history of this language that allows us to understand the workings of our universe.

History can be broadly separated into two parts: prehistory and recorded history. In general, prehistory is the period in which no written records of any kind were left behind. History becomes much more exact when written records can be examined. The earliest written remains are those of tally marks found on animal bones or horns. The most famous is the Ishango bone found in the Congo in 1960. This baboon bone shows groups of parallel gashes that many scholars believe are a form of tally system, perhaps even a lunar calendar. The bone also has a quartz crystal placed in the end for making the markings.

So the earliest written record left to us by man is in the form of a mathematical artifact. One of the earliest acts Homo sapiens performed that started us on the road on which we are still traveling is the combining of abstraction and language into the science and art of mathematics.

Unlike any other area of study, mathematics is the study of truths. Mathematical styles and methods may change, but mathematical truths are eternal, and how these truths are discovered is the story of the history of mathematics. Seltman reflects upon this in her apology for Harriot's *Praxis* [105]:

> First, I am going to ask the (perhaps surprising) questions: in what sense does mathematics have a history? I ask this question because there is one sense in which it does not have a real history, and that resides in its consisting of permanent truths such as the theorems of Euclid, under the conditions of Euclidean axioms, which express unchanging relationships. Correct mathematics remains correct however out of date it is. Archimedes' methods for finding volumes and areas may be obsolete but are still valid relative to the axioms of his system. This is the unchanging aspect of mathematics. In this sense, mathematics has no history—it just persists through time. It is not a changing subject. Remember that mathematics itself consists only of *ideas* (relationships between elements)—it is not the symbols, or the paper they are written on or the mathematician who has constructed the mathematics. What does have a history is the total process, consisting of the mathematical ideas together with the mathematician who constructed them, together with the notation, symbolism, methodology, and so on and all the other connections to the social and individual milieu that constitutes the developing mathematical process. And this certainly does have a history. But the mathematical ideas *per se* do not. They are pure, noetic relationships, which are permanently true. We must, therefore, distinguish the mathematics in itself from the historical process as a whole, which includes all the external connections that are contingent to it. Such contingent relationships are 'external' to the ideas to which they are related and may or may not be absorbed into the process, thus altering it. The thought processes of mathematicians, or other people connecting in some way to the mathematics, are contingencies in relation to it and may change the existing mathematics or not in some way, thus being the engine for the historical (or developmental) process of mathematical growth. It is the 'may or may not be' property of contingency which renders history of any sort unpredictable. In the light of all this, there is a second general theoretical question to be asked and that is: Are there revolutions or fundamental changes in mathematics as there are said to be in science? ... There are surely no revolutions in mathematics in itself, since mathematics, consisting of pure relational ideas, may become obsolete but is not replaced (in the sense of being overthrown) by changes. It remains correct, no matter what. There are, however, revolutions in the process of mathematical development, which includes the mathematician and his/her thinking and all the external relations of the mathematics in itself. And such revolutions may occur with regard to, say, symbolism, axioms, methodology, and so forth. All this was stated above.

Studying the history of mathematics is an intriguing and inspiring journey. Along the way you will discover people and places and ways of thinking about numbers that are different from the mathematical concepts you have encountered thus far. You will find that there is no "one way" to do mathematics, but in fact there are many ways: each way is different and creative, evolved from the needs of mankind and the social and historical context of the time. The history of mathematics is an important story to tell and to learn because mathematics is the most "human" of all the human endeavors.

We wish you well on your journey through time as you explore the history of mathematics.

Amy Shell-Gellasch and J. B. Thoo
July 2015

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