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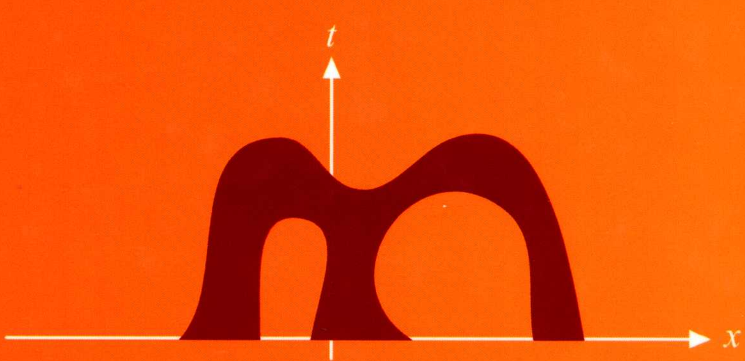
Nonlinear Partial Differential Equations

Asymptotic Behavior of Solutions and Self-Similar Solutions



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In memory of Professor Tetsuro Miyakawa
– with our profound admiration

Preface

The purpose of this book is to present typical methods (including rescaling methods) for the examination of the behavior of solutions of nonlinear partial differential equations of diffusion type. For instance, we examine such equations by analyzing special so-called self-similar solutions. We are in particular interested in equations describing various phenomena such as the Navier–Stokes equations. The rescaling method described here can also be interpreted as a renormalization group method, which represents a strong tool in the asymptotic analysis of solutions of nonlinear partial differential equations. Although such asymptotic analysis is used formally in various disciplines, not seldom there is a lack of a rigorous mathematical treatment. The intention of this monograph is to fill this gap. We intend to develop a rigorous mathematical foundation of such a formal asymptotic analysis related to self-similar solutions. A self-similar solution is, roughly speaking, a solution invariant under a scaling transformation that does not change the equation. For several typical equations we shall give mathematical proofs that certain self-similar solutions asymptotically approximate the typical behavior of a wide class of solutions.

Since nonlinear partial differential equations are used not only in mathematics but also in various fields of science and technology, there is a huge variety of approaches. Moreover, even the attempt to cover only a few typical fields and methods requires many pages of explanations and collateral tools so that the approaches are self-contained and accessible to a large audience. It is not our intention to survey many topics of nonlinear partial differential equations. Our aim in this book is to explain some asymptotic methods by studying typical examples.

Historically, partial differential equations were introduced soon after the notion of differentiation and integration was settled, with the purpose to model dynamical behavior of the motion of bodies such as a string or a membrane. A partial differential equation (PDE) is an equation describing a functional relation of a set of unknowns and their derivatives. Here the unknowns depend in general on several independent variables such as time and space. If the unknowns depend only on one variable, the equation is called an ordinary differential equation (ODE). Thus, compared with ODEs there is a much larger diversity of problems modeled by PDEs. In fact, various PDEs are proposed to model phenomena not only in physics, for example in mechanics, electromagnetics, and thermodynamics, but also in various other fields of science and technology such as social sciences and finance. On the other hand, PDEs do not only describe real-world phenomena, but also play an important role in the description of mathematical objects such as those, for example, in differential geometry and complex analysis. If a PDE is linear with respect to the unknowns and their derivatives, it is called a linear partial differential equation. Typical examples of linear PDEs are the heat equation, the Poisson equation, and the Laplace equation in electromagnetics. However, in the modeling of certain phenomena there appear several key PDEs that are not linear. PDEs of this type are called nonlinear partial differential equations. A typical example is given by the Navier–Stokes equations, which represent the fundamental equations of hydrodynamics. There is a huge variety of nonlinear PDEs, and so far it seems impossible to discuss fundamental problems in a unified way. Typical problems in mathematical analysis include a solvability problem—existence of solutions of a PDE—under suitable supplemental conditions such as initial or boundary conditions. For linear PDEs such problems can be discussed somewhat in a unified way. This, however, seems to be hopeless for the nonlinear case, since each nonlinear PDE has a special structure. So, we do not intend to establish a unified theory at the present stage. Rather we mostly study a specific class of nonlinear PDEs having a similar structure. (Note that the set of linear PDEs is a special class of PDEs.) Even for fundamental problems such as solvability, necessary prerequisites depend upon equations. From the applied point of view other problems such as profile and behavior of solutions, are also very important. Indeed, researchers in applied fields often conjecture the behavior of solutions by studying special solutions. However, there is a tendency among mathematical books treating PDEs in a rigorous way to spend many pages on solvability problems, and it is often difficult to explain the behavior of solutions.

The aim of this book is to study the behavior of solutions in a rigorous way by discussing typical examples without even assuming knowledge of functional analysis. For this purpose, the structure of this book differs essentially from the setup of usual mathematical textbooks. In the conventional style, authors explain fundamental universal theory for PDE analysis, such as elementary functional analysis, and discuss PDEs in that framework. This is a smart way to encode a lot of mathematical information in a small number of pages, which is also very efficient. In this book, however, we pursue a different way. We study directly the behavior of solutions of particular equations without preparing the fundamental theory. Instead, we discuss fundamental tools used in the analysis of these PDEs in the second part of this book. We hope that the reader will learn to deal with tools such as calculus inequalities during the study of PDEs. This more direct way should give students a strong motivation for the study of such fundamental tools and an idea of their usefulness for applications.

The book at hand consists of two parts. Part I includes Chapters 1, 2, and 3. Part II includes Chapters 4, 5, 6, and 7. In Part I we present a way to study the behavior of solutions of nonlinear PDEs of diffusion type using self-similar solutions. In Chapter 1 we show as a preliminary result by two methods that the large-time behavior of solutions of the heat equation is asymptotically self-similar. The first method relates to a representation formula of the solution. This argument is simple; however, it is restrictively applicable to nonlinear PDEs. The second method replaces the problem by the task of showing the convergence of a family of functions of rescaled solutions. This argument, however, applies to a wide range of problems.

In fact, in Chapter 2 we analyze in detail by the second method the two-dimensional vorticity equations (obtained from the Navier–Stokes equations). We shall prove that the vorticity, which is the solution of the vorticity equations, is asymptotically self-similar as time tends to infinity. Moreover, its behavior is proportional to the behavior of the Gauss kernel (also called the Gaussian vortex), provided that the total circulation is small. We present a proof that is more transparent than the ones given in the previous literature and that is based on an improvement of the fundamental $L^q - L^1$ estimate (Section 2.3) for the heat equation with transport term. We also complete the proof by giving an estimate (Section 2.5.2) for the family of rescaled functions (which is missing in the literature). Our purpose is to get a sharp result with a method as elementary as possible. For example, the estimates on the derivatives of the vorticity (Section 2.4.2) are new in the sense that they include the cases $p = 1$ and $p = \infty$. The proof is elementary in the sense that it does not use a complicated function space or interpolation of spaces.

As an application of the asymptotic behavior of the vorticity we discuss in Section 2.6 the formation of the Burgers vortex in three dimensions. A few years ago the convergence to the Gaussian vortex was proved without assuming that the total circulation is small. We include this beautiful result, which is based on relative entropy, in Section 2.8. In order to make this book self-contained we also give a proof of all key statements (except for the lemma in Section 2.5.2), including those in Part II by admitting the unique solvability of the vorticity equations as well as the solvability of the heat equation with transport term. We hope that the reader, while following the proofs, will learn about the significance of the calculus inequalities, provided in Chapter 6, in the analysis of these individual PDEs. Almost all inequalities invoked in Chapters 1 and 2 are proved in Part II, unless their proof is given in Chapters 1 and 2 already.

In Chapter 3 we first present a typical result of large-time asymptotic behavior of solutions for the porous medium equation, however, without giving a proof. Then, we present a method to analyze asymptotic behavior of solutions for the mean curvature flow equations near a singularity. These equations are often used to model the motion of phase boundaries such as antiphase grain boundaries. We show that the key monotonicity formula is also valid for the harmonic map flow equation and the semilinear heat equation. Furthermore, we give an elementary proof (Section 3.2.3) of the uniqueness of self-similar solutions of the mean curvature flow equations for axisymmetric surfaces. Finally, as an example of non-diffusion-type equations we mention a nonlinear Schrödinger equation and a generalized KdV equation. Also for these equations we present an existence result of self-similar solutions describing large-time behavior and behavior near a singularity, respectively. Here we just state the results without giving a proof. So, Chapter 3 is a collection of several different topics, while Chapter 2 is written toward one explicit goal.

In Part II we give explicit proofs for various important functional analytic statements invoked in Part I. In Chapter 4 we prove decay estimates for the heat equation and uniqueness of the solution, if the initial value is given by the Dirac delta distribution. We review several basic notions, such as the fundamental solution for the heat equation with transport term, and prove its unique existence. For the reader's convenience we give also a proof of integration by parts in unbounded multidimensional domains. In Chapter 5 we give a variant of the Ascoli–Arzelà theorem, which is a fundamental compactness result for families of functions. This variant applies also to families defined on a domain that is not necessarily compact. In Chapter 6 we prove several important inequalities. Except for the boundedness of singular integral operators, we present proofs based on estimates for the solution of the heat equation. Compared to other existing textbooks this approach is quite unusual. From these interesting applications we learn that estimates for the solution of the heat equation can be important in various situations, although they are rather elementary. Our intention is not to give the shortest proof. We rather try to explain variants of the proofs. In Chapter 7, we summarize basic knowledge on integration theory and on bounded linear operators.

The inequalities in Chapter 6 are very important in the analysis of nonlinear PDEs in general, i.e., also for PDEs not treated in this book. In mathematical analysis it is often crucial how to estimate various quantities. These inequalities are presented rather in textbooks on real analysis than in textbooks on PDEs. Even though these inequalities are classical results, we included their proofs in order to make this book self-contained. We often mention unsolved problems at the present stage in italics in order to animate further research. (In fact, a problem raised in the Japanese version published in 1999 has been solved.) In the approaches presented in Part I and Part II we usually proceed as follows: first we state what we want to show and discuss applications; then we give the technical details of the proof. We hope that the reader will be able to read results and proofs with a clear view why the corresponding problems are studied, although some of them look just technical. We also remark that the range of the topics treated in this book is too broad to give a complete list of references. We therefore just tried to give a list of typical references. However, we included "notes and comments" or "research history" in some chapters, which should help the reader to find further related literature. To shorten the description we often refer to a theorem, proposition, lemma, corollary, remark, or definition in a particular subsection just by its subsection number. For example, instead of writing "the theorem in §2.2.1" we often write "Theorem 2.2.1" if no confusion seems likely.

It is widely known that nonlinear analysis is significant for science and technology. As a very attractive topic, the analysis of nonlinear PDEs can be regarded as an important subfield of nonlinear analysis. However, to understand nonlinear PDEs in a rigorous mathematical way, it is often believed that a wide-ranging knowledge including Lebesgue integration theory, functional analysis, theory of distributions, real analysis, and the theory of ODEs is necessary. Of course, if one is familiar with these subjects, the description of results can be simplified and their treatment can be unified in an elegant way (in contrast to the approach presented in this book, where we tried not to use these theories). However, some readers might be interested in studying properties of solutions of nonlinear PDEs as soon as possible (before mastering these prerequisites). This book is written mainly for such readers. The layout is chosen in a way that the reader will gain necessary analytic knowledge and intuition naturally during the study of the behavior of solutions of PDEs. For this purpose several elementary facts such as differentiation under the integral sign are elaborately explained in Part II. As a consequence this requires a great deal of text on linear PDEs (although this is also useful in analyzing nonlinear PDEs). Very nonlinear structure is discussed mainly in Chapter 3.

The prerequisite to read Part I is only calculus including integration by parts in higher dimensions. If one reads Part II in a logically complete way, an elementary part of Lebesgue integration theory is necessary. Our hope is that the reader will see how mathematical theory taught in freshman and sophomore courses represents the basis for various theories with beautiful applications to PDEs.

For the reader who is interested in large-time asymptotic behavior of solutions of the heat and vorticity equations we suggest to read Sections 1.1, 2.1, 2.2, 2.6, 2.7.1, 2.8 first. For the reader who is interested in various applications of self-similar solutions we suggest to read Section 2.7.3 and Chapter 3. We hope these sections are useful to readers who are also interested in various other disciplines than mathematics such as, for instance, hydrodynamics and engineering.

The authors are grateful to Professor Haim Brezis for inviting them to write this book and for his patience.

The present book is based on the first two authors' book *Hisenkei Henbibun Hoteishiki* published in Japanese by Kyoritsu Shuppan in 1999. The book is not just a simple translation of the Japanese version. We expanded and revised several parts. However, the structure and the spirit are similar to the Japanese version.

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March 2010

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Yoshikazu Giga
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Part II Useful Analytic Tools

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