天元基金影印数学丛书

Analysis II

分析 II（影印版）

Roger Godement

高等教育出版社
HIGHER EDUCATION PRESS

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序言

为了更好地借鉴国外数学教育与研究的成功经验，促进我国数学教育与研究事业的发展，提高高等学校数学教育教学质量，本着“为我国热爱数学的青年创造一个较好的学习数学的环境”这一宗旨，天元基金赞助出版“天元基金影印数学丛书”。

该丛书主要包含国外反映近代数学发展的纯数学与应用数学方面的优秀书籍，天元基金邀请国内各个方向的知名数学家参与选题的工作，经专家遴选、推荐，由高等教育出版社影印出版。为了提高我国数学研究生教学的水平，暂把选书的目标确定在研究生教材上。当然，有的书也可作为高年级本科生教材或参考书，有的书则介于研究生教材与专著之间。

欢迎各方专家、读者对本丛书的选题、印刷、销售等工作提出批评和建议。

天元基金领导小组

2007 年 1 月

郑重声明

Contents

V – Differential and Integral Calculus

§ 1. The Riemann Integral

The theory of integration expounded in this Chapter dates from the XIX$^{\text{th}}$ century; it was, and remains, of great use in classical mathematics, and its simplicity has rewarded all who have written for beginners in the subject. For professional mathematicians it has been dethroned by the much more powerful, and in some respects simpler, theory invented by Henri Lebesgue around 1900, and perfected in the course of the first half of the XX$^{\text{th}}$ century by dozens of others; we present a small part of it in the Appendix to this Chapter. The "Riemann" theory expounded in this Chapter therefore has only a pedagogic interest.

1 – Upper and lower integrals of a bounded function

Let us first recall the definitions of Chap. II, n° 11.

A scalar (i.e. complex-valued) function φ defined on a compact, or more generally, bounded, interval I is said to be a *step function* if one can find a partition (Chap. I) of I into a finite number of intervals I_k such that φ is constant on each I_k; no conditions are imposed on the I_k. Such a partition will be said to be *adapted to* φ.

When $I = (a, b)$ this is the same as requiring the existence of a finite sequence of points of I satisfying

$$a = x_1 \leq x_2 \leq \ldots \leq x_{n+1} = b \tag{1.1}$$

and such that φ is constant on each *open* interval $]x_k, x_{k+1}[$, because the values it takes at a point x_k have no connection with those it takes to the right or left of this point, and are irrelevant to the calculation of traditional integrals[1].

A sequence of points satisfying (1) is called a *subdivision* of the interval I. A subdivision by the points y_h is said to be *finer* than the subdivision (1) when the x_k appear among the y_h, in other words when the second subdivision is obtained by subdividing each of the component intervals in (1). The definition is similar for two partitions (I_k) and (J_h) of I: the second is said to be finer than the first if every J_h is contained in one of the I_k, in other words if the second partition of I is obtained by partitioning each of the I_k themselves into intervals (namely, those J_h contained in I_k).

If $\varphi(x) = a_k$ for every $x \in I_k$ one calls the number

$$m(\varphi) = \sum a_k m(I_k) = \sum \varphi(\xi_k) m(I_k) \tag{1.2}$$

the *integral of* φ *over* I, where, for every interval $J = (u, v)$, the number $m(J) = v - u$ denotes the length or *measure* of J, and where ξ_k is any point of I_k. Since the I_k of zero measure do not matter in (2) one can replace the partition by a subdivision (1) and write

$$m(\varphi) = \sum \varphi(\xi_k)(x_{k+1} - x_k) \qquad \text{with } x_k < \xi_k < x_{k+1} \tag{1.3}$$

since φ is constant, so equal to $\varphi(\xi_k)$, on $]x_k, x_{k+1}[$.

Since there are infinitely many ways of choosing the I_k – every finer partition, for example, will equally be adapted to calculating the integral –, we have to show that the sum (2) does not depend on the choice of the I_k. So let (J_h) be another partition of I into intervals such that $\varphi(x) = b_h$ for every $x \in J_h$. Since each I_k is the union of the pairwise disjoint intervals $I_k \cap J_h$, as is shown by the relation

$$X = X \cap I = X \cap \bigcup J_h = \bigcup X \cap J_h,$$

valid for every subset X of I, we have

$$m(I_k) = \sum_h m(I_k \cap J_h)$$

and similarly

$$m(J_h) = \sum_k m(I_k \cap J_h)$$

where, by convention, $m(\emptyset) = 0$. Thus

$$\sum a_k m(I_k) = \sum a_k m(I_k \cap J_h), \tag{1.4}$$

$$\sum b_h m(J_h) = \sum b_h m(I_k \cap J_h), \tag{1.5}$$

[1] This is not the same in generalisations of the classical theory. See n° 30.

where, on the right hand sides, we sum over all the pairs (k, h). We thus have only to prove that

$$m(I_k \cap J_h) \neq 0 \text{ implies } a_k = b_h,$$

which is clear: on $I_k \cap J_h$, which is nonempty since its length is not zero, the function φ is equal simultaneously to a_k and to b_h.

This argument shows immediately that

$$m(\lambda\varphi + \mu\psi) = \lambda m(\varphi) + \mu m(\psi) \tag{1.6}$$

for any step functions φ and ψ and constants λ and μ: consider partitions (I_k) and (J_h) of I adapted to φ and ψ, write a_k for the value of φ on I_k and b_h for that of ψ on J_h, and calculate the integrals of φ, ψ and $\lambda\varphi + \mu\psi$ using the intervals $I_k \cap J_h$ on which φ, ψ and $\lambda\varphi + \mu\psi$ are equal respectively to a_k, b_h and $\lambda a_k + \mu b_h$; in effect we are adding the relations (4) and (5), multiplied respectively by λ and μ, term-by-term.

Since it is clear that the integral of a positive function (i.e. one whose values are all positive) is positive, we see that

$$\varphi \leq \psi \qquad \text{implies} \qquad m(\varphi) \leq m(\psi) \tag{1.7}$$

for real-valued φ and ψ, since $m(\psi) - m(\varphi) = m(\psi - \varphi) \geq 0$ by (6) and $\psi - \varphi$ is positive.

Finally, the triangle inequality applied to (2) shows that

$$|m(\varphi)| \leq \sum |\varphi(\xi_k)| m(I_k) = m(|\varphi|) \leq \sum \|\varphi\|_I m(I_k)$$

always, where, as in Chap. III, n° 7, we write in a general way that

$$\|f\|_I = \sup_{x \in I} |f(x)|.$$

Since $\sum m(I_k) = m(I)$ we finally obtain the inequality

$$|m(\varphi)| \leq m(|\varphi|) \leq m(I)\|\varphi\|_I. \tag{1.8}$$

This completes the "theory" of integration as it applies to step functions. It rests on two properties of lengths which are the starting point for all later generalisations:

(M 1): the measure of an interval is positive;
(M 2): measure is additive, i.e. if an interval J is the union of a finite number of pairwise disjoint intervals J_k then $m(J) = \sum m(J_k)$.

There are many other interval-functions which have these properties. One can, for example, choose a continuous function $\mu(x)$ which is increasing in the wide sense on I and put[2]

[2] For an arbitrary increasing function one has to take account of its discontinuities and modify the formula to obtain a reasonable theory. See n° 32 on Stieltjes measures.

$$\mu(J) = \mu(v) - \mu(u) \quad \text{if } J = (u, v).$$

One can also take a finite or countable set $D \subset I$ and assign to each $\xi \in D$ a "mass" $c(\xi) > 0$, with $\sum c(\xi) < +\infty$, and then put

$$\mu(J) = \sum_{\xi \in J} c(\xi)$$

for every interval J, so that the measure of a singleton interval can very well be > 0; in this example property (M 2) reduces to the associativity formula for absolutely convergent series. We obtain *discrete measures* in this way.

For a "measure" μ satisfying (M 1) and (M 2) the integral of a step function is, by definition, the number $\mu(\varphi)$ given by the formula (2), replacing the letter m by the letter μ. For a discrete measure, one clearly finds that $\mu(\varphi) = \sum c(\xi)\varphi(\xi)$, summing over all the $\xi \in D$. These generalisations will be studied at the end of this chapter, but the reader may be interested to observe, every time we use the traditional integral, those results which depend only on the properties (M 1) and (M 2) of "Euclidean" or "Archimedean" measure, or, as one now calls it, of "Lebesgue measure" (since it was for this that Lebesgue constructed his grand integration theory) because these properties extend to the general case. Certain results which, on the contrary, use the explicit construction starting from the usual measure, mainly concern the relations between integrals and derivatives, Fourier series and integrals, partial differential equations, almost all applications to physical sciences, etc. They rest on an obvious though fundamental property of the usual measure: it is invariant under translation; see below, (2.20).

Now let us pass on to arbitrary *bounded* real functions on a bounded interval I (in general compact).

Given a bounded real-valued function f on I there exist step functions, even constant functions, φ and ψ, such that $\varphi \leq f \leq \psi$, i.e. $\varphi(x) \leq f(x) \leq \psi(x)$ for every $x \in I$. By (7) we must have $m(\varphi) \leq m(\psi)$, and every reasonable definition of $m(f)$ must satisfy $m(\varphi) \leq m(f) \leq m(\psi)$. We therefore examine the *lower* and *upper* integrals of f over I defined by the formulae

(1.9) $$m_*(f) = \sup_{\varphi \leq f} m(\varphi), \qquad m^*(f) = \inf_{\psi \geq f} m(\psi)$$

where φ and ψ range over the sets of step functions such that $\varphi \leq f \leq \psi$.

As we have seen in Chap. II, n° 11, we have $m_*(f) \leq m^*(f)$ since every number $m(\varphi)$ is less than the $m(\psi)$, so is less than their lower bound $m^*(f)$, which, larger than all the $m(\varphi)$, is also larger than their upper bound $m_*(f)$. Since the constant functions equal to $-\|f\|_I$ and $+\|f\|_I$ feature among the functions φ and ψ respectively, we even have

(1.10) $$-m(I)\|f\|_I \leq m_*(f) \leq m^*(f) \leq m(I)\|f\|_I.$$

Relation (6) does not extend to the lower and upper integrals of arbitrary functions; if it did, the theory of integration would finish with n° 2 of this chapter. However, we always have the inequalities

$$(1.11) \quad m_*(f+g) \geq m_*(f) + m_*(g), \qquad m^*(f+g) \leq m^*(f) + m^*(g).$$

Among the step functions less than $f+g$ are the sums $\varphi + \psi$, where φ is less than f and where ψ is less than g; consequently, $m_*(f+g)$ is greater than all the numbers of the form $m(\varphi + \psi) = m(\varphi) + m(\psi)$. It remains to note that if A and B are two sets of real numbers, and if one writes $A + B$ for the set of numbers $x + y$ where $x \in A$ and $y \in B$, then

$$\sup(A+B) = \sup(A) + \sup(B)$$

with a similar relation for the lower bounds (exercise!), so that every number larger than the $x+y$ is larger than $\sup(A)+\sup(B)$. Whence the first relation (11). The second is proved in the same way, reversing the inequalities.

It is easier to show that

$$(1.12) \quad m_*(cf) = cm_*(f), \qquad m^*(cf) = cm^*(f) \qquad \text{for every } c \geq 0$$

and

$$(1.13) \qquad m_*(-f) = -m^*(f);$$

it is enough to note that multiplication by -1 transforms the step functions below f into those above $-f$.

2 – Elementary properties of integrals

The most natural definition of integrable functions with real values is that they should satisfy the condition

$$m^*(f) = m_*(f),$$

the common value of the two sides then being the value of the integral $m(f)$ of f; one extends the definition to functions $f = g + ih$ with complex values by requiring both g and h to be integrable and putting

$$m(f) = m(g) + im(h).$$

This definition, adopted in the First French Edition for reasons of simplicity, has several drawbacks; in particular, it is not obvious — although, of course, true — that the absolute value $|f| = [\text{Re}(f)^2 + \text{Im}(f)^2]^{\frac{1}{2}}$ of a complex-valued integrable function is again integrable, as Michel Ollitrault, a reader of the First Edition, has justly remarked to me. We shall therefore abandon this definition temporarily, to recover it later, and we shall adopt a method used in the modern theory too. We shall develop it for complex-valued functions, but it will also apply to functions with values in a finite dimensional vector space, or even a Banach space, which is not the case for the first simplistic definition.

We shall say that a function f is *integrable* if, for any $r > 0$, there is a step function φ (with values in the same space as f if one is integrating vector-valued functions) such that

$$m^*(|f-\varphi|) < r. \tag{2.1}$$

If f has real values this means, intuitively, that the *numerical* (and not algebraic) measure of the area in the plane included between the graphs of f and φ is $< r$; there is no point in assuming φ "above" or "below" f. It comes to the same to require the existence of a sequence of step functions φ_n such that

$$\lim m^*(|f-\varphi_n|) = 0 \tag{2.1'}$$

or, as one says, which *converges in mean* to f. One says "in mean" because the fact that the upper integral of a positive function $h(x)$ is very small does not prevent h from taking very large values on very small intervals: $10^{100}10^{-200} = 10^{-100}$.

To define the integral of an integrable function f one uses the relation (1'). By the triangle inequality we have

$$|\varphi_p - \varphi_q| \leq |\varphi_p - f| + |f - \varphi_q|$$

and so

$$|m(\varphi_p) - m(\varphi_q)| = |m(\varphi_p - \varphi_q)| \leq m^*(|\varphi_p - f|) + m^*(|f-\varphi_q|),$$

by (1.11). The sequence with general term $m(\varphi_n)$ therefore satisfies Cauchy's convergence criterion (Chap. III, n° 10, Theorem 13). Its limit depends only on f. For if ψ_n is another sequence of step functions satisfying (1') the relation

$$|\varphi_n - \psi_n| \leq |f - \varphi_n| + |f - \psi_n|$$

shows, in a similar way, that $m(\varphi_n) - m(\psi_n)$ tends to 0.

It is natural to call the limit of the $m(\varphi_n)$ (common to all sequences of step functions converging to f in mean) the *integral* of f, and to denote it by $m(f)$. This kind of argument, used in many other places, is similar to the one we used to define a^x for $a > 0$ and $x \in \mathbb{R}$, by approximating x by a sequence of rational numbers x_n and showing that the sequence a^{x_n} converges to a limit independent of x (Chapter IV, § 1, end of n° 2).

If an integrable function f has real (resp. positive) values then its integral is real (resp. positive). If f is real, and if in (1') one replaces φ_n by Re φ_n one decreases the function $|f - \varphi_n|$ and so its upper integral, so that the sequence of real functions $\mathrm{Re}(\varphi_n)$ again converges to f in mean, whence the first result. If, moreover, f is positive, in which case one may assume the φ_n real, one argues in the same way, replacing the $\varphi_n(x)$ by 0 on the intervals where $\varphi_n < 0$: this can only decrease the value of $|f(x) - \varphi_n(x)|$, and so of the upper integral.

If f and g are integrable then $f + g$ is integrable and

$$m(f+g) = m(f) + m(g).$$

Take step functions φ_n and ψ_n converging in mean to f and g, write

$$|(f+g) - (\varphi_n + \psi_n)| \leq |f - \varphi_n| + |g - \psi_n|$$

to show that $\varphi_n + \psi_n$ converges to $f + g$ in mean, and use (1.6).

If f is integrable then so is αf for any $\alpha \in \mathbb{C}$, and $m(\alpha f) = \alpha m(f)$. Obvious: multiply f and φ by α in (1) and apply (1.12).

These first results already show, for real integrable f and g, that

$$f \leq g \text{ implies } m(f) \leq m(g),$$

since $0 \leq m(g - f) = m(g) + m(-f) = m(g) - m(f)$.

If f is integrable then so is $|f|$, and

$$|m(f)| \leq m(|f|) \leq m(I)\, \|f\|_I \tag{2.2}$$

where, we recall, $\|f\|_I = \sup |f(x)|$ is the norm of uniform convergence on I (Chap. III, n° 7). For any complex numbers α and β we have $\big||\alpha| - |\beta|\big| \leq |\alpha - \beta|$, whence, in the notation of (1'),

$$\big||f(x)| - |\varphi_n(x)|\big| \leq |f(x) - \varphi_n(x)| \qquad \text{for all } x \in I$$

and so $m^*(|f| - |\varphi_n|) \leq m^*(|f - \varphi_n|)$; this proves that $|f|$ is integrable like f, since the $|\varphi_n|$ are also step functions. Since the integrals of φ_n and $|\varphi_n|$ converge to those of f and $|f|$, by definition of the latter, and since (2) applies to the φ_n, one obtains the first inequality (2) in the limit. The second follows from the fact that $|f(x)| \leq \|f\|_I$ everywhere on I, so that $m(|f|)$ is less than the integral of the constant function $x \mapsto \|f\|_I$.

The complex-valued function f is integrable if and only if the functions $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are. If so,*

$$m(f) = m[\mathrm{Re}(f)] + i.m[\mathrm{Im}(f)].$$

Since $|\mathrm{Re}(f) - \mathrm{Re}(\varphi_n)| \leq |f - \varphi_n|$, with a similar relation for the imaginary parts, it is clear that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are integrable if f is; the relation to be shown then follows from the linearity properties already obtained; these show no less trivially that f is integrable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are.

A real function f is integrable if and only if $m^*(f) = m_*(f)$.

Suppose first that $m_*(f) = m^*(f)$. Then, for every $r > 0$ there are step functions φ and ψ framing f whose integrals are equal to within r. Since $|f - \psi| = f - \psi \leq \varphi - \psi$ it follows that $m^*(|f - \psi|) \leq m(\varphi - \psi) < r$, whence the integrability of f.

Suppose conversely that f is integrable and consider a step function φ such that

$$m^*(|f - \varphi|) < r;$$

one may assume φ real as above. Since $m^*(|f - \varphi|)$ is, by definition, the lower bound of the numbers $m(\psi)$ over all step functions $\psi \geq |f - \varphi|$, the strict inequality proves the existence of a step function ψ such that

$$|f - \varphi| < \psi \qquad \& \qquad m(\psi) < r.$$

Since $\varphi - \psi \leq f \leq \varphi + \psi$ we have thus framed f between two step functions whose difference has integral $\leq 2r$; so $m^*(f) = m_*(f)$. Moreover,

$$m(\varphi - \psi) \leq m^*(f) \leq m(\varphi + \psi);$$

since f is integrable we already know that this relation is preserved if one replaces $m^*(f)$ by $m(f)$, whence $m(f) = m^*(f)$, since the extreme terms in the preceding relation are equal to within $2r$.

To sum up:

Theorem 1. *Let I be a bounded interval. (i) If the bounded functions f and g are integrable on I, then so likewise is $\alpha f + \beta g$ for any constants α and β, and*

$$m(\alpha f + \beta g) = \alpha m(f) + \beta m(g). \tag{2.3}$$

(ii) If f is defined, bounded and integrable on I, then the function $|f|$ is integrable, and

$$|m(f)| \leq m(|f|) \leq m(I)\|f\|_I = m(I)\sup|f(x)|. \tag{2.4}$$

(iii) The integral of a positive function is positive.

The standard notation

$$m(f) = \int_I f(x)dx$$

will be explained later (n° 3).

The definition of integrable functions shows immediately that, on a *compact* interval, *every regulated function is integrable*; for every $r > 0$ there exists, by the definition (Chap. III, n° 12) a step function φ such that $|f(x) - \varphi(x)| < r$ for every x; then, by (1.10), $m^*(|f - \varphi|) < m(I)r$, whence the result. We shall prove later (n° 7) that, on a compact interval, every continuous function is regulated, so integrable. One hardly needs more subtle results in elementary analysis.

It is not difficult to construct non-integrable functions: it is enough to take the Dirichlet function $f(x)$ on I, equal to 0 if $x \in \mathbb{Q}$ and to 1 if $x \notin \mathbb{Q}$. Now, if a step function $\varphi \leq f$ is constant on the intervals I_k of a partition of I, it must be ≤ 0 on every nonsingleton I_k since such an interval contains rational numbers where $f(x) = 0$; likewise, every step function $\psi \geq f$ must be "almost" everywhere ≥ 1. Thus $m_*(f) = 0$ and $m^*(f) = m(I)$. The Lebesgue theory allows one to integrate the function f, with the same result as if one had $f(x) = 1$ everywhere, and this because $\mathbb{Q}$ is countable. It may appear bizarre to consider such functions – Newton would have said that one does not meet them in Nature[3] –, but it is one of those which led Cantor towards his great set theory, not to be confused with the trivialities of Chap. I. Even though the function in question is strange, one cannot deny it the merit of simplicity; if analysis is incapable of integrating such functions, one might begin to suspect that this is the fault of analysis and not of the function ...

We said above that the integral of a positive function is positive; could it perhaps be zero? This is one of the fundamental questions which the complete Lebesgue theory allows one to resolve. For the moment we make just two elementary remarks.

If the integral of a continuous positive *function f is zero, then $f = 0$.* For if we have $f(a) = r > 0$ for some $a \in I$, then the continuity of f shows that $f(x) > r/2$ on an interval $J \subset I$ of length > 0; if φ is the step function equal to $r/2$ on J and to 0 elsewhere then $m(f) \geq m(\varphi) = r\,m(J)/2 > 0$.

This result (which presupposes the integrability of the continuous functions and uses the fact that, in the traditional theory, the measure of a nonempty open interval is > 0) does not extend to discontinuous functions. For a positive step function for example, it is clear that the integral vanishes if and only if the points where the function does not vanish are finite in number. In the much more general case of a regulated function, the apposite condition is that the set defined by the relation $f(x) \neq 0$ should be *countable* (n° 7).

Before stating the next theorem let us note that if we have real functions f and g defined on any set X we can construct the functions

$$\begin{aligned} \sup(f,g) &: \quad x \mapsto \max[f(x), g(x)], \\ \inf(f,g) &: \quad x \mapsto \min[f(x), g(x)]; \end{aligned}$$

these definitions generalise in the obvious way to a finite number of functions (and even to an infinite number on replacing max and min by sup and inf) and

[3] We will meet them in computer science when there exist machines capable of distinguishing the rational numbers automatically from the others.