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Shi Zhongci Wang Ming

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Preface to the Series in Information and Computational Science

Since the 1970s, Science Press has published more than thirty volumes in its series Monographs in Computational Methods. This series was established and led by the late academician, Feng Kang, the founding director of the Computing Center of the Chinese Academy of Sciences. The monograph series has provided timely information of the frontier directions and latest research results in computational mathematics. It has had great impact on young scientists and the entire research community, and has played a very important role in the development of computational mathematics in China.

To cope with these new scientific developments, the Ministry of Education of the People's Republic of China in 1998 combined several subjects, such as computational mathematics, numerical algorithms, information science, and operations research and optimal control, into a new discipline called Information and Computational Science. As a result, Science Press also reorganized the editorial board of the monograph series and changed its name to Series in Information and Computational Science. The first editorial board meeting was held in Beijing in September 2004, and it discussed the new objectives, and the directions and contents of the new monograph series.

The aim of the new series is to present the state of the art in Information and Computational Science to senior undergraduate and graduate students, as well as to scientists working in these fields. Hence, the series will provide concrete and systematic expositions of the advances in information and computational science, encompassing also related interdisciplinary developments.

I would like to thank the previous editorial board members and assistants, and all the mathematicians who have contributed significantly to the monograph series on Computational Methods. As a result of their contributions the monograph series achieved an outstanding reputation in the community. I sincerely wish that we will extend this support to the new Series in Information and Computational Science, so that the new series can equally enhance the scientific development in information and computational science in this century.

Shi Zhongci
2005.7

Preface

The finite element method has achieved a great deal of success in many fields since it was first suggested in the structural analysis in the fifth decade of last century. Today it is a powerful numerical tool solving partial differential equations. The scholars in our country contributed much to the foundation and development of finite element method. Feng's work is original, independent of the West, to the foundation of the finite element method.

The basic idea of the finite element method is using discrete solutions on finite element spaces to approximate the continuous solutions on infinite dimensional space V according to the variational principle. The typical steps of constructing finite element spaces are the following.

(1) The domain Ω, the continuous solution defined on, is subdivided into some subdomains, which are called elements.

(2) On each element, an m-dimensional polynomial space and m nodal parameters are selected, such that each polynomial in the space is determined uniquely by a group of nodal parameters. The function values and derivatives at some points on the element are often taken as the nodal parameters.

(3) A piecewise polynomial space V_h on domain Ω, called finite element space, is obtained by linking the nodal parameters on elements in some way.

For the mathematical foundation of the finite element method, there is a well-known result:

The approximation of finite element solution to the real solution is dependent on the approximation of finite element space V_h to the space V, provided V_h is a subspace of V.

The approximate property of the finite element spaces can be dealt with by the interpolation theory of the finite elements.

When finite element space V_h is a subspace of V, the method is called conforming or standard finite element method. In general, the conforming property of finite element space implies, for the $2m$-th order elliptic boundary value problems, that V_h is a subspace of $C^{m-1}(\overline{\Omega})$. So the problems are sometimes called C^{m-1} problems. There are a lot of conforming finite elements for the boundary value problems of second order elliptic partial differential equations, including the element with the least number of nodal parameters. There are also some conforming finite elements for the fourth order problems, which have more numbers of nodal parameters. For example, at least 18 parameters and polynomials with a degree not greater than 5 are required when the elements are triangles. In the three dimensional case, Zenicek constructed a conforming tetrahedral finite element with 9-th degree of polynomials and 220 nodal parameters. This offers serious computational difficulty, either the dimension of the finite element space is fairly large or the structure of the finite element space is complicated. It is interested how to decrease the number of nodal parameters and the degree of polynomials.

The first approach is to subdivide the element into some small elements and to use lower degree piecewise polynomials on the element such that the continuous requirement is satisfied. The finite element space obtained by this way is also complicated and loses the simple feature of finite element method. So this method is not really useful.

A successful approach is to relax the C^{m-1} continuity on Ω. The elements obtained in this way are called nonconforming finite elements. Some nonconforming elements, such as the Wilson element, the Morley element, are convergent, while some ones, for example, the two dimensional Zienkiewicz element, are only convergent for the special subdivision of Ω. It became very important what are the convergence conditions for the nonconforming elements. Irons proposed a method, so called 'Patch Test', from the numerical practice. The idea of the patch test is base on that the element can solve the problem accurately for the case of constant strain. The patch test is very simple and popular in engineers. But the patch test, without additional conditions, is neither necessary nor sufficient. A mathematical condition, namely the generalized patch test, was suggested by Stummel. It is a necessary and sufficient condition. Shi checked the convergence of some elements by the generalized patch test, while proposed a more really useful condition: the F-E-M Test.

Although it has been proved by mathematicians that the patch test is neither necessary nor sufficient, engineers still believe that it is a necessary and sufficient condition for the convergence of the nonconforming elements, provided certain condition such as the weak continuity. Fortunately, Wang has shown that the patch test becomes sufficient with certain assumptions, in addition, if an element does not pass the patch test then there exist subdivisions which make the element divergent.

Besides the nonconforming finite element method, there is the hybrid finite element method for relaxing the C^{m-1} continuity. The method takes the continuity on the interelement boundaries as the constraint, and introduces the Lagrangian multiplier functions on the interelement boundaries to relax the continuity constraint. Then the original variational problem becomes new one related to the subdivision of Ω. The finite element method based on the new variational problem is the hybrid finite element method. For a hybrid finite element, a nonconforming finite element can be reduced from it. In this sense, the hybrid finite element may be viewed as a special case of the nonconforming finite element.

Either the patch test or the generalized patch test is only an analysis tool for the convergence of finite elements. How to construct good elements is another problem. Generally, the function values and the derivative values at vertices of elements are preferable, with a view to simple construction and small total degrees of freedom. But it is difficult to ensure certain continuity on the interelement boundaries, such as the Zienkiewicz element. To guarantee the convergence, in two-dimensional case, for example, the integral mean values of normal derivative on the edges of elements and the function values on midpoints of the edges are selected, such as the Veubake element. It leads to the increase of the total degrees of freedom.

To find better elements, some construction methods were suggested by mechanists, such as, the quasi-conforming element method proposed by Tang et al, the generalized conforming element method by Long, the free formula method and the energy-orthogonal method by Bergan et al. Some elements constructed by these methods have very good convergence. The convergence of the quasi-conforming element method was proved by Zhang et al, and one of the others by Shi et al.

The methods of the nonconforming element, the quasi-conforming element, the generalized conforming element, the free formula method and the energy-orthogonal methods, are all different from conforming element method. Hence these methods are called non-standard finite element methods. To analyze these non-standard finite element methods, the method of approximation with multiple sets of function (AMSF method) was suggested by Zhang, and the double set parameter method (DSP method) was suggested by Shi and Chen. These mathematical tools established the mathematical theory of the non-standard finite element methods. This book will introduce the DSP method. For AMSF method, the readers can find it in the book of Zhang and Wang.

There is another way to overcome the C^{m-1} difficulty, that is the mixed finite element method. The method changes the original partial differential equation into a system of some equations of lower order, for example, the biharmonic equation into a system of two equations of second order, then considers the finite element approximation to the new problem. Although the final scheme of the finite elements may be different from the one obtained directly from the original problem, it is nothing but a finite element method for the new problem. In this point of view, the mixed finite element method itself is not nonstandard. Some nonstandard finite elements for the new problem can also be used.

The subject of the finite element method and its mathematical theory are very broad, the finite element method is applied to a wide class of scientific computations. This book deals with the mathematical theory of the standard and non-standard finite element methods for the boundary value problems of elliptic partial differential equation. The topics of the book are restricted to

(1) the mathematical descriptions,

(2) the convergent conditions,

(3) a priori and a posteriori error estimates of finite element solutions,

(4) the basic properties of finite element spaces,

about these finite element methods. The book should serve as an introduction to these aspects.

The book is divided into twelve chapters. Chapter 1 considers the variational principle of the elliptic boundary value problems. Some preliminaries are presented, such as, the essential results about the Sobolev spaces, the variational forms of the boundary value problems of the Poisson equation and the biharmonic equation, the Lax-Milgram Lemma about the abstract variational problems, the Ritz method and the Galerkin method and their error estimates.

Chapter 2 is devoted to the basic aspects of finite elements — how to construct a finite element, the corresponding interpolation operator and the finite element spaces. Some examples of conforming or nonconforming finite elements for the elliptic boundary value problems are presented. From the view of the construction method for finite elements and finite element spaces and of the error analysis for the interpolation error, the conforming element and the nonconforming elements are the same. In addition, they have no difference in the algorithm realization. Therefore, they are listed together in this book.

Chapter 3 is arranged to discuss the interpolation theory of finite elements. The affine technique about the finite elements, which is popular in the discussion of the approximation of finite elements, is introduced. The interpolation error estimates, the inverse inequality and the approximate error of the finite element spaces are given. The interpolation result in more general case is also discussed.

Chapter 4 considers the conforming finite element method for the elliptic boundary value problem. The convergence and error estimate of the finite element solutions for the second and fourth order problems are discussed, while the Aubin-Nistche technique is included. The a posteriori error estimate of the finite element solutions for the $2m$-th order problem is given as well.

Chapter 5 and Chapter 6 are devoted to the mathematical theory of the nonconforming finite element methods. The nonconforming finite element methods for the second, fourth and $2m$-th order elliptic problems, together with their a priori and a posteriori error estimates, are presented in Chapter 5. The convergent conditions of the nonconforming element methods, such as the patch test, the generalized patch test, the weak patch test and the F-E-M test, are discussed in Chapter 6. The superapproximation of the finite elements and the strange convergent behavior of some nonconforming elements are also discussed in Chapter 6.

Chapter 7 introduces the quasi-conforming finite element method. The basic idea of the method, the examples of the method, the convergent conditions, a priori and a posteriori error estimate are given.

Chapter 8 is devoted to the unconventional finite element method, such as the free formula method and the energy-orthogonal method. Their convergent analysis and error estimate, a priori and a posteriori, are included.

Chapter 9 introduces the DSP method. Some non-standard finite element methods, for example the generalized conforming element method, can be dealt with by the DSP method. The first two sections discuss the description of DSP method, the well posed conditions and its convergence. The next two sections are its applications to the non-standard finite element methods and the construction of new finite elements. The a posteriori error estimate is described finally.

Chapter 10 considers the properties of the non-standard finite element spaces. These properties are the analogues of the embedding and compact properties which the Sobolev spaces have. In addition, some inequalities on the spaces, such as the generalized Poincare-Friedrichs inequality and the generalized Korn inequality, are presented.

The last two chapters deal with the L^∞ error estimates of the finite elements for the two-dimensional Poisson equation and the plate bending problem. Both conforming method and non-standard method are under consideration.

There is a huge amount of literature about the finite element method and its mathematical theory. The references listed in the book are a very small part. Either these references are the sources of the materials of the book or they have been referred to in preparing the present book.

The motivation writing this book was the suggestion of Mr. Lin Peng of Science Press. The first manuscript of the book was finished in June, 1996, while the second author was visiting Department of Applied Mathematics of the Hong Kong Polytechnic University as a visitor of the Croucher Foundation. The final version added some new materials. The first six chapters of the book had been taught in the course of 'Finite Element Methods II' for the graduate students in School of Mathematical Sciences, Peking University.

The authors would like to acknowledge Mr. Lin Peng, the Croucher Foundation, Department of Applied Mathematics and Office of Academic and Professional Collaboration of the Hong Kong Polytechnic University. Much thanks to the colleagues there, especially to Professor S. H. Hou, Professor T. M. Shih, Professor C. B. Liem and Ms. Annie Lam. Many thanks to the graduate students of School of Mathematical Sciences, Peking University, who read the manuscript of the book and made valuable suggestions, particularly to Mr. Gao Boran.

Contents

Chapter 1
Variational Principle

The finite element method is base on the variational formula of partial differential equation. The paper [1] of Feng's work about the finite element method was just entitled by 'A difference formulation based on the variational principle'. For the variational problem corresponding to the partial differential equation, there are two classical approximate methods, the Ritz method and the Galerkin method. The finite element method is obtained when the finite element spaces are used as the finite dimensional spaces in the Ritz method or in the Galerkin method. The knowledge about the Sobolev spaces is preliminary to the mathematical theory of finite element method. Section 1.1 will give the essential results about Sobolev spaces without proof. Section 1.2 and Section 1.3 will introduce the variational forms of boundary value problems of the Poisson equation and the biharmonic equation respectively. Section 1.4 will consider the Lax-Milgram lemma about the abstract variational problems. The Galerkin method and the Ritz method of the variational problems will be described in the last section.

The material of Section 1.1 comes from Adams' book [2]. The one of Section 1.2 and Section 1.3 comes from [3]. The content of the last two sections, except Theorem 1.5.2, comes from Ciarlet's book [4].

1.1 Sobolev Space

Let $\boldsymbol{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ denote the point in n dimensional space $\mathbf{R}^n$. Let Ω be a bounded and connected domain in $\mathbf{R}^n$ with boundary $\partial\Omega$. When m is a nonnegative integer or ∞, let $C^m(\Omega)$ be the set consisting of all m-th order continuously differentiable functions in Ω, $C^m(\overline{\Omega})$ the set consisting of all m-th order continuously differentiable functions on the closed domain $\overline{\Omega}$, and $C_0^m(\Omega)$ the set consisting of all functions in $C^m(\Omega)$ with their compact support in Ω. When $m = 0$, the superscript m can be omitted, i.e., $C^0(\Omega), C^0(\overline{\Omega})$ and $C_0^0(\Omega)$ can be denoted by $C(\Omega), C(\overline{\Omega})$ and $C_0(\Omega)$ respectively.

A vector $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ with its components nonnegative integers is called a multi-index. Define $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$. The partial derivative operator can be written as

$$\partial^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Let

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right)^{\mathrm{T}}.$$

When m is a nonnegative integer, the linear space $C^m(\overline{\Omega})$ is a Banach space with the following norm,

$$\|v\|_{C^m(\overline{\Omega})} = \max_{|\boldsymbol{\alpha}| \leqslant m} \max_{\boldsymbol{x} \in \overline{\Omega}} |\partial^{\boldsymbol{\alpha}} v(\boldsymbol{x})|, \quad \forall v \in C^m(\overline{\Omega}).$$

To describe the Sobolev spaces, the definition of the generalized derivatives is required. Let u be a locally integrable function on Ω in Lebesgue's sense. If there exists another function v locally integrable on Ω such that,

$$\int_\Omega v\phi \,\mathrm{d}\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u \partial^{\boldsymbol{\alpha}} \phi \,\mathrm{d}\boldsymbol{x}, \quad \forall \phi \in C_0^\infty(\Omega),$$

then v is called a generalized derivative of u, denoted by $\partial^{\boldsymbol{\alpha}} u$. Obviously, the generalized derivatives are the generalization of the classical derivatives.

For $\varrho \in [1, \infty]$ and a nonnegative integer m, define

$$W^{m,\varrho}(\Omega) = \{ \, u \in L^\varrho(\Omega) \mid \partial^{\boldsymbol{\alpha}} u \in L^\varrho(\Omega), \ |\boldsymbol{\alpha}| \leqslant m \, \}.$$

Then $W^{m,\varrho}(\Omega)$ is a Banach space with the Sobolev norm $\|\cdot\|_{m,\varrho,\Omega}$ given as follows, for $u \in W^{m,\varrho}(\Omega)$,

$$\|u\|_{m,\varrho,\Omega} = \left(\sum_{|\boldsymbol{\alpha}| \leqslant m} \int_\Omega |\partial^{\boldsymbol{\alpha}} u|^\varrho \,\mathrm{d}\boldsymbol{x} \right)^{1/\varrho}, \quad 1 \leqslant \varrho < \infty,$$

$$\|u\|_{m,\infty,\Omega} = \max_{|\boldsymbol{\alpha}| \leqslant m} \operatorname*{esssup}_{\boldsymbol{x} \in \Omega} |\partial^{\boldsymbol{\alpha}} u(\boldsymbol{x})|.$$

$W^{m,\varrho}(\Omega)$ is called Sobolev space. When $\varrho = 2$, $W^{m,\varrho}(\Omega)$ is denoted by $H^m(\Omega)$, its norm by $\|\cdot\|_{m,\Omega}$. $H^m(\Omega)$ is a Hilbert space. The following theorem is essential in the theory of the Sobolev space.

Theorem 1.1.1 *If the boundary $\partial\Omega$ of domain Ω is Lipschitz continuous and $1 \leqslant \varrho < \infty$, then $C^\infty(\overline{\Omega})$ is dense in $W^{m,\varrho}(\Omega)$.*

Denote the closure of space $C_0^\infty(\Omega)$ in norm $\|\cdot\|_{m,\varrho,\Omega}$ by $W_0^{m,\varrho}(\Omega)$, then it is a Banach space. When $\varrho = 2$, it is denoted by $H_0^m(\Omega)$, and it is also a Hilbert space.