Graduate Texts in Mathematics

128

Partial Differential Equations

Springer-Verlag

Jeffrey Rauch

Partial Differential Equations

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Preface

This book is based on a course I have given five times at the University of Michigan, beginning in 1973. The aim is to present an introduction to a sampling of ideas, phenomena, and methods from the subject of partial differential equations that can be presented in one semester and requires no previous knowledge of differential equations. The problems, with hints and discussion, form an important and integral part of the course.

In our department, students with a variety of specialties—notably differential geometry, numerical analysis, mathematical physics, complex analysis, physics, and partial differential equations—have a need for such a course.

The goal of a one-term course forces the omission of many topics. Everyone, including me, can find fault with the selections that I have made.

One of the things that makes partial differential equations difficult to learn is that it uses a wide variety of tools. In a short course, there is no time for the leisurely development of background material. Consequently, I suppose that the reader is trained in advanced calculus, real analysis, the rudiments of complex analysis, and the language of functional analysis. Such a background is not unusual for the students mentioned above. Students missing one of the "essentials" can usually catch up simultaneously.

A more difficult problem is what to do about the Theory of Distributions. The compromise which I have found workable is the following. The first chapter of the book, which takes about nine fifty-minute hours, does not use distributions. The second chapter is devoted to a study of the Fourier transform of tempered distributions. Knowledge of the basics about $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$, $\mathscr{D}'(\Omega)$, and $\mathscr{E}'(\Omega)$ is assumed at that time. My experience teaching the course indicates that students can pick up the required facility. I have provided, in an appendix, a short crash course on Distribution Theory. From Chapter 2 on, Distribution Theory is the basic language of the text, providing a good setting for reinforcing the fundamentals. My experience in teaching this course is that students have less difficulty with the distribution theory than with geometric ideas from advanced calculus (e.g. $d\varphi$ is a one-form which annihilates the tangent space to $\{\varphi = 0\}$).

There is a good deal more material here than can be taught in one semester. This provides material for a more leisurely two-semester course and allows the reader to browse in directions which interest him/her. The essential core is the following:

Chapter 1. Almost all. A selection of examples must be made.

Chapter 2. All but the L^p theory for $p \neq 2$. Some can be left for students to read.

Chapter 3. The first seven sections. One of the ill-posed problems should be presented.

Chapter 4. Sections 1, 2, 5, 6, and 7 plus a representative sampling from Sections 3 and 4.

Chapter 5. Sections 1, 2, 3, 10, and 11 plus at least the statements of the standard Elliptic Regularity Theorems.

These topics take less than one semester.

An introductory course should touch on equations of the classical types, elliptic, hyperbolic, parabolic, and also present some other equations. The energy method, maximum principle, and Fourier transform should be used. The classical fundamental solutions should appear. These conditions are met by the choices above.

I think that one learns more from pursuing examples to a certain depth, rather than giving a quick gloss over an enormous range of topics. For this reason, many of the equations discussed in the book are treated several times. At each encounter, new methods or points of view deepen the appreciation of these fundamental examples.

I have made a conscious effort to emphasize qualitative information about solutions, so that students can learn the features that distinguish various differential equations. Also the origins in applications are discussed in conjunction with these properties. The interpretation of the properties of solutions in physical and geometric terms generates many interesting ideas and questions.

It is my impression that one learns more from trying the problems than from any other part of the course. Thus I plead with readers to attempt the problems.

Let me point out some omissions. In Chapter 1, the Cauchy–Kowaleskaya Theorem is discussed, stated, and much applied, but the proof is only indicated. Complete proofs can be found in many places, and it is my opinion that the techniques of proof are not as central as other things which can be presented in the time gained. The classical integration methods of Hamilton and Jacobi for nonlinear real scalar first-order equations are omitted entirely. My opinion is that when needed these should be presented along with symplectic geometry. There is a preponderance of linear equations, at the expense of nonlinear equations. One of the main points for nonlinear equations is their differences with the linear. Clearly there is an order in which these things should be learned. If one includes the problems, a reasonable dose of nonlinear examples and phenomena are presented. With the exception of the elliptic theory, there is a strong preponderance of equations with constant coefficients, and especially Fourier transform techniques. The reason for this choice is that one can find detailed and interesting information without technical complexity. In this way one learns the ideas of the theory of partial differential equations at minimal cost. In the process, many methods are introduced which work for variable coefficients and this is pointed out at the appropriate places.

Compared to other texts with similar level and scope (those of Folland, Garabedian, John, and Treves are my favorites), the reader will find that the present treatment is more heavily weighted toward initial value problems. This, I confess, corresponds to my own preference. Many time-independent problems have their origin as steady states of such time-dependent problems and it is as such that they are presented here.

A word about the references. Most are to textbooks, and I have systematically referred to the most recent editions and to English translations. As a result the dates do not give a good idea of the original publication dates. For results proved in the last 40 years, I have leaned toward citing the original papers to give the correct chronology. Classical results are usually credited without reference.

I welcome comments, critiques, suggestions, corrections, etc. from users of this book, so that later editions may benefit from experience with the first.

So many people have contributed in so many different way to my appreciation of partial differential equations that it is impossible to list and thank them all individually. However, specific influences on the structure of this book have been P.D. Lax and P. Garabedian from whom I took courses at the level of this book; Joel Smoller who teaches the same course in a different but related way; and Howard Shaw whose class notes saved me when my own lecture notes disappeared inside a moving van. The integration of problems into the flow of the text was much influenced by the *Differential Topology* text of Guillemin and Pollack. I have also benefited from having had exceptional students take this course and offer their criticism. In particular, I would like to thank Z. Xin whose solutions, corrections, and suggestions have greatly improved the problems. Chapters of a preliminary version of this text were read and criticized by M. Beals, J.L. Joly, M. Reed, J. Smoller, M. Taylor, and M. Weinstein. Their advice has been very helpful. My colleagues and coworkers in partial differential equations have taught me much and in many ways. I offer a hearty thank you to them all.

The love, support, and tolerance of my family were essential for the writing of this book. The importance of these things to me extends far beyond professional productivity, and I offer my profound appreciation.

Contents

CHAPTER 1

Power Series Methods

§1.1. The Simplest Partial Differential Equation

It takes a little time and a few basic examples to develop intuition. This is particularly true of the subject of partial differential equations which has an enormous variety of technique and phenomena within its confines. This section describes the simplest nontrivial partial differential equation

$$u_t(t, x) + cu_x(t, x) = 0, \qquad t, x \in \mathbb{R}, \quad c \in \mathbb{C}. \tag{1}$$

The equation is of first order, is linear with constant coefficients, and involves derivatives with respect to both variables. The unknown is a possibly complex valued function u of two real variables. This example reveals one of the fundamental dichotomies of the subject, the equation is hyperbolic if $c \in \mathbb{R}$ and elliptic otherwise. The equation is radically different in these two cases in spite of the similar appearance.

The use of "t" is meant to suggest time. One can use the equation to march forward in time as follows. Given u at time t, $u(t, \cdot)$, one can compute the value of

$$u_t(t, \cdot) = -c\partial_x u(t, \cdot),$$

and then advance the time using

$$u(t + \Delta t, \cdot) \approx u(t, \cdot) + u_t(t, \cdot)\Delta t = (1 - c\Delta t \partial_x)u(t, \cdot). \tag{2}$$

This marching algorithm suggests that the *initial value problem* or *Cauchy problem* is appropriate. Thus, given $g(x)$ we seek u satisfying (1) and the initial condition

$$u(0, \cdot) = g(\cdot). \tag{3}$$

For $g \in C^\infty(\mathbb{R})$ and $n \in \mathbb{N}$ we may choose a time step $\Delta t = 1/n$ and find approximate values

$$u\left(\frac{k}{n}, \cdot\right) \approx \left(1 - \frac{c}{n}\partial_x\right)^k g(\cdot). \tag{4}$$

Since the approximation (2) improves as Δt decreases to zero it is not unreasonable to think that as $n, k \to \infty$ with $k/n = t$ fixed, the approximations on the right approach the values $u(t, \cdot)$ of a solution.

With $t = k/n$, (4) reads

$$u(t, \cdot) \approx \left(1 - \frac{tc\partial_x}{k}\right)^k g(\cdot). \tag{5}$$

Letting k tend to infinity suggests the formal identity

$$u(t, \cdot) = \exp(-ct\partial_x)g. \tag{6}$$

For polynomial g, formally expanding the exponential and using Taylor's Theorem yields

$$u(t, \cdot) = \sum \frac{(-ct)^n g^n(\cdot)}{n!} = g(\cdot - ct). \tag{7}$$

It is easy to verify that for polynomial g, $g(x - ct)$ is indeed a solution of the initial value problem and is also the limit of the approximations (4). In fact, if g is the restriction to $\mathbb{R}$ of an analytic function on $|\text{Im } x| < R$, then one has convergence for $|t| < R/|c|$ to the solution $g(x - ct)$.

If c is real, then the formula $g(x - ct)$ still provides a solution even when g does not have an analytic continuation to a neighborhood of the real axis. However, the approximations (5) will not converge if the derivatives of g grow faster than those of an analytic function.

Finally, if c is complex then the formula suggests that g must have a natural extension from real to complex values of x in order for there to be a solution.

The ideas suggested by the formal computations are next verified by examining the initial value problem (1), (3) following a different and easier route.

For real c, the differential equation (1) asserts that the directional derivative of u in the direction $(1, c)$ vanishes (Figure 1.1.1). Thus $u \in C^1(\mathbb{R}^2)$ is a solution if and only if u is constant on each of the lines $x - ct = \text{constant}$. These lines, integral curves of the vector field $\partial/\partial t + c\partial/\partial x$, are called *characteristic lines* or *rays*. This observation yields the following result.

Theorem 1. *If c is real and $g \in C^1(\mathbb{R})$, there is a unique solution $u \in C^1(\mathbb{R}^2)$ to the initial value problem* (1), (3). *The solution is given by the formula $u(t, x) = g(x - ct)$. If $g \in C^k(\mathbb{R})$ with $k > 1$, then $u \in C^k(\mathbb{R})$.*

The solution u represents undistorted wave propagation with speed c. The characteristic lines have slope dt/dx equal to $1/c$ and speed dx/dt equal to c. The value of u at $\bar{t}, \bar{x}$ is determined by g at $\bar{x} - c\bar{t}$. This illustrates the ideas of *domain of determinacy* and *domain of influence*. The domain of determinacy of $\bar{t}, \bar{x}$ is the point $(0, \bar{x} - c\bar{t})$ on the line $t = 0$. The domain of influence of the point $(0, \underline{x})$ on the initial line is the characteristic $x - ct = \underline{x}$ (Figure 1.1.2).

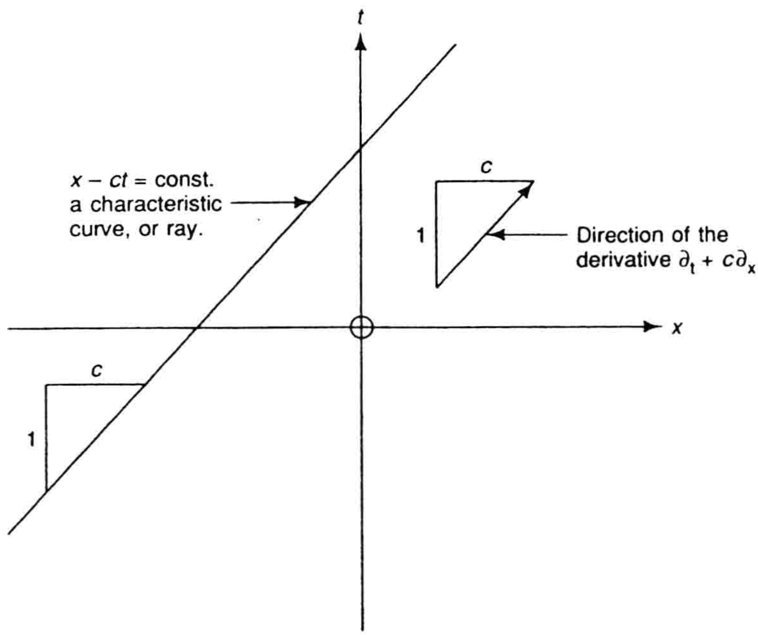

Figure 1.1.1

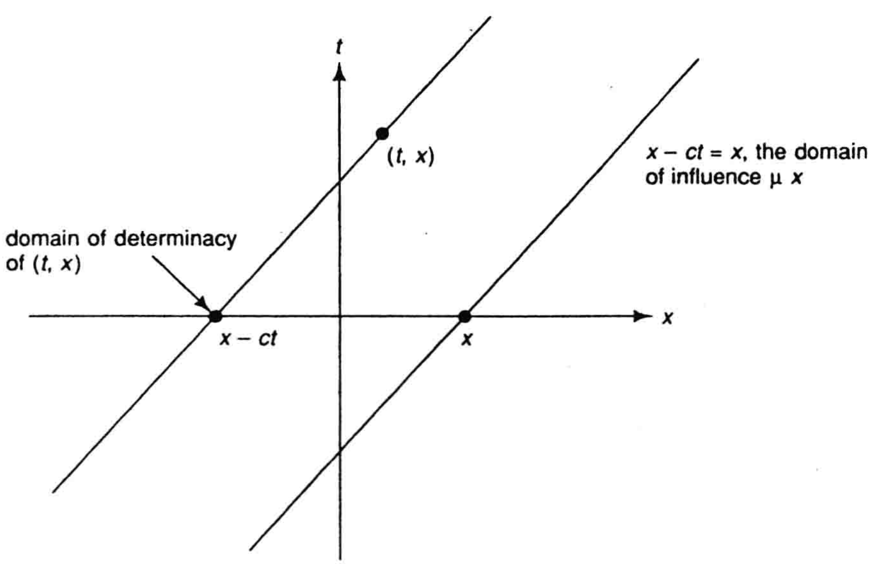

Figure 1.1.2

Nearby initial data g yield nearby solutions u. A precise statement is that the map from g to u is continuous from $C^k(\mathbb{R})$ to $C^k(\mathbb{R}^2)$ for any $k \geq 1$. The topology in the spaces C^k are defined by a countable family of seminorms. To avoid this complication at this time, consider data g which belong to $BC^k(\mathbb{R})$, the set of C^k functions each of whose derivatives, of order less than or equal to k, is bounded on $\mathbb{R}$. This is a Banach space with norm

$$\|g\|_{BC^k(\mathbb{R})} \equiv \sum_{j \leq k} \left\| \left(\frac{d}{dx}\right)^j g \right\|_{L^\infty(\mathbb{R})}.$$

$BC^k(\mathbb{R}^d)$ is defined similarly with

$$\|u\|_{BC^k(\mathbb{R}^d)} \equiv \sum_{j_1+\cdots+j_d \leq k} \left\| \left(\prod \left(\frac{\partial}{\partial x_k}\right)^{j_k}\right) u \right\|_{L^\infty(\mathbb{R}^d)}.$$

For the solution of the initial value problem (1), (3)

$$\partial_t^j \partial_x^k u(t, x) = (-c)^j \left(\frac{d}{dx}\right)^{j+k} g(x - ct).$$

An immediate consequence is the following corollary.

Corollary 2. *For $c \in \mathbb{R}$ and $k \geq 1$ in $\mathbb{N}$, the map from the Cauchy data g to the solution u of the initial value problem* (1), (3) *is continuous from $BC^k(\mathbb{R})$ to $BC^k(\mathbb{R}^2)$.*

The case of imaginary c is quite different. In particular, the initial value problem is no longer well set. We analyze the case $c = -i$, leaving the case of general $c \in \mathbb{C}\backslash\mathbb{R}$ to the problems.

Suppose $\Omega \subset \mathbb{R}^2$ is open and $u \in C^1(\Omega)$ satisfies $u_t = iu_x$. Identify Ω with a subset of $\mathbb{C}$ by

$$x, t \mapsto x + it.$$

The corresponding subset of $\mathbb{C}$ is denoted $\Omega_{\mathbb{C}}$. Define a function $f : \Omega_{\mathbb{C}} \to \mathbb{C}$ by

$$f(x + it) \equiv u(t, x).$$

With $z \equiv x + it$, we have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x},$$

$$\frac{\partial f}{\partial t} = \frac{\partial u}{\partial t} = i\frac{\partial u}{\partial x},$$

so

$$\frac{\partial f}{\partial t} = i\frac{\partial f}{\partial x}. \tag{8}$$

Equation (8) is called the *Cauchy–Riemann equation*. In elementary function theory one shows that the solutions, called *holomorphic* or *analytic functions* of $z = x + it$, are infinitely differentiable. Moreover, if $p \in \Omega_{\mathbb{C}}$, then f is equal to the sum of a convergent power series in $z - p$,

$$f = \sum_{n=0}^{\infty} a_n (z - p)^n, \qquad |z - p| < \operatorname{dist}(p, \partial\Omega_{\mathbb{C}}).$$

Differentiating term by term shows that the converse is also true, that is, convergent power series in $z - p$ are solutions.

Theorem 3. *$u \in C^1(\Omega)$ satisfies $u_t - iu_x = 0$ if and only if it defines a holomorphic function on $\Omega_{\mathbb{C}}$.*

Next consider the initial value problem $u_t - iu_x = 0$, $u(0, \cdot) = g$. If there is a solution on a neighborhood of $(0, \underline{x})$, then u is holomorphic. Thus

$$u = \sum a_n (z - \underline{x})^n,$$

so

$$g = \sum a_n (x - \underline{x})^n,$$

is given by a convergent power series. Such a function is called *real analytic*. Conversely, if g is real analytic at $\underline{x}$ then the above formula defines u holomorphic near $\underline{x} + i0$.

Warning. If the a_n are complex, such real analytic functions need not be real valued. They are defined on a real domain, hence the name.

Theorem 4. *The initial value problem*

$$u_t + iu_x = 0, \qquad u(0, \cdot) = g(\cdot),$$

has a C^1 solution on a neighborhood of $(0, \underline{x})$ if and only if g is the restriction to $\mathbb{R}$ of a holomorphic function defined on a neighborhood of $\underline{x}$, that is, if and only if g is real analytic at x.

As a consequence, we see that if g is C^∞ but not real analytic at $\underline{x}$, then the approximation scheme 2, 4 cannot converge to a solution of the initial value problem on a neighborhood of $(0, \underline{x})$. It is not difficult to show that one does have convergence for real analytic g.

Summary

(i) For $c \in \mathbb{R}$, the initial value problem is nicely solvable.

(ii) For $c \in \mathbb{C}\backslash\mathbb{R}$, $u_t + cu_x = 0$ has only real analytic solutions. The initial value problem is not solvable unless the data are real analytic.

(iii) For $c \in \mathbb{R}$ the equation is *hyperbolic*. For $c \in \mathbb{C}\backslash\mathbb{R}$, it is *elliptic*. These terms will be defined later and describe two of the most important classes of partial differential equations.

PROBLEMS

1. If $c \in \mathbb{C}\backslash\mathbb{R}$, $g \in C^1(\mathbb{R})$, then the initial value problem
$$u_t + cu_x = 0, \qquad u(0, x) = g(x),$$
has a C^1 solution on a neighborhood of the origin if and only if g is real analytic on a neighborhood of the origin.

2. Prove that if $c \in \mathbb{R}$ and u is a $C^1(\mathbb{R}^2)$ solution of the equation $\partial_t u + c\partial_x u = 0$, then
$$\{(t, x) \in \mathbb{R}^2 : u \in C^k \text{ on a neighborhood of } t, x\}$$
is a union of rays.
DISCUSSION. This elementary result is typical. Solutions of partial differential equations inherit a great deal of structure from the equation they satisfy. This result asserts *propagation of singularities* and *propagation of regularity* along rays.

3. Prove that if $u \in C^\infty(\mathbb{R}^2)$ satisfies $\partial_t u + c\partial_x u = 0$ with c real, and k is a nonnegative integer, then
$$\{(t, x) : u \text{ vanishes to order } k \text{ at } (t, x)\}$$
is a union of rays. For any closed set $\Gamma \subset \mathbb{R}^2$ which is a union of rays, prove that there is a u as above such that Γ is exactly the set where u vanishes.
DISCUSSION. Contrast this to the case where c is not real. Then, if a solution vanishes on any open set it must vanish identically.

4. Show that for $c \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^2)$ there is one and only one solution to the initial value problem
$$u_t + cu_x = f, \qquad u(0, x) = 0.$$
Find a formula for the solution. Find an $f \in C^1(\mathbb{R}^2)$ such that the solution is not in $C^2(\mathbb{R}^2)$.
DISCUSSION. This may be surprising since "first derivatives in C^1 indicate $u \in C^2$". However, the partial differential equation contains only a linear combination of first derivatives. Nevertheless, when $c \in \mathbb{C}\backslash\mathbb{R}$ the equation is elliptic and, in a sense, controls all derivatives. In that case, $u_t + cu_x \in C^\infty$ implies that $u \in C^\infty$. Also u has one more derivative than $u_t + cu_x$, but not in the sense of the classical spaces C^k (see Propositions 2.4.5 and 5.9.1, and Problem 5.9.3).

5. For the nonlinear initial value problem,
$$u_t + cu_x + u^2 = 0, \qquad u(0, x) = g(x), \qquad c \in \mathbb{R},$$
show that if $g \in C_0^\infty(\mathbb{R})$, g not identically zero, there is a local solution $u \in C^\infty(\{-\delta < t < \delta\} \times \mathbb{R})$ but that the solution does not extend to a C^∞ solution on all of $\mathbb{R}^2$.
DISCUSSION. This blow-up of solutions is just like that for the nonlinear ordinary differential equation $dy/dt = y^2$. Nonlinear partial differential equations have more subtle blow-up mechanisms too. See the formation of shocks discussed in §1.9.