Mathematical
Surveys
and
Monographs

Volume 195

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2010 *Mathematics Subject Classification*. Primary 11G15, 14K02, 14L05, 14K15, 14D15.

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Library of Congress Cataloging-in-Publication Data

Chai, Ching-Li, author.
Complex multiplication and lifting problems / Ching-Li Chai, Brian Conrad, Frans Oort.
pages cm — (Mathematical surveys and monographs ; volume 195)
Includes bibliographical references and index.
ISBN 978-1-4704-1014-8 (alk. paper)
1. Multiplication, Complex. 2. Abelian varieties. I. Conrad, Brian, 1970– author. II. Oort, Frans, 1935– author. III. Title.

QA564.C44 2014
516.3′53—dc23

2013036892

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10 9 8 7 6 5 4 3 2 1 19 18 17 16 15 14

Complex Multiplication and Lifting Problems

This book is dedicated to

John Tate

for what he taught us, and for inspiring us

Preface

During the *Workshop on Abelian Varieties* in Amsterdam in May 2006, the three authors of this book formulated two refined versions of a problem concerning lifting into characteristic 0 for abelian varieties over a finite field. These problems address the phenomenon of *CM lifting*: the lift into characteristic 0 is required to be a CM abelian variety (in the sense defined in 1.3.8.1). The precise formulations appear at the end of Chapter 1 (see 1.8.5), as problems (I) and (IN).

Abelian surface counterexamples to (IN) were found at that time; see 2.3.1–2.3.3, and see 4.1.2 for a more thorough analysis. To our surprise, the same counterexamples (typical among *toy models* as defined in 4.1.3) play a crucial role in the general solution to problems (I) and (IN). This book is the story of our adventure guided by CM lifting problems.

Ching-Li Chai thanks Hsiao-Ling for her love and support during all these years. He also thanks Utrecht University for hospitality during many visits, including the May 2006 Spring School on Abelian Varieties which concluded with the workshop in Amsterdam. Support by NSF grants DMS 0400482, DMS 0901163, and DMS120027 is gratefully acknowledged.

Brian Conrad thanks the many participants in the "CM seminar" at the University of Michigan for their enthusiasm on the topic of complex multiplication, as well as Columbia University for its hospitality during a sabbatical visit, and gratefully acknowledges support by NSF grants DMS 0093542, DMS 0917686, and DMS 1100784.

Frans Oort thanks the University of Pennsylvania for hospitality and stimulating environment during several visits.

We are also grateful to Burcu Baran, Bas Edixhoven, Ofer Gabber, Johan de Jong, Bill Messing, Ben Moonen, James Parson, René Schoof, and Jonathan Wise for insightful and memorable discussions.

Contents

Introduction

I restricted myself to characteristic zero: for a short time, the quantum jump to $p \neq 0$ was beyond the range ... but it did not take me too long to make this jump.

—Oscar Zariski

The arithmetic of abelian varieties with complex multiplication over a number field is fascinating. However this will not be our focus. We study the theory of complex multiplication in mixed characteristic.

Abelian varieties over finite fields. In 1940 Deuring showed that an elliptic curve over a finite field can have an endomorphism algebra of rank 4 [**33**, §2.10]. For an elliptic curve in characteristic zero with an endomorphism algebra of rank 2 (rather than rank 1, as in the "generic" case), the j-invariant is called a *singular j-invariant.* For this reason elliptic curves with even more endomorphisms, in positive characteristic, are called *supersingular.*[1]

Mumford observed as a consequence of results of Deuring that for any elliptic curves E_1 and E_2 over a finite field κ of characteristic $p > 0$ and any prime $\ell \neq p$, the natural map

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}(E_1, E_2) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(\overline{\kappa}/\kappa)]}(T_\ell(E_1), T_\ell(E_2))$$

(where on the left side we consider only homomorphisms "defined over κ") is an isomorphism [**118**, §1]. The interested reader might find it an instructive exercise to reconstruct this (unpublished) proof by Mumford. Tate proved in [**118**] that the analogous result holds for all abelian varieties over a finite field and he also incorporated the case $\ell = p$ by using p-divisible groups. He generalized this result into his influential conjecture [**117**]:

> *An ℓ-adic cohomology class*[2] *that is fixed under the Galois group should be a $\mathbb{Q}_\ell$-linear combination of fundamental classes of algebraic cycles when the ground field is finitely generated over its prime field.*

Honda and Tate gave a classification of isogeny classes of simple abelian varieties A over a finite field κ (see [**50**] and [**121**]), and Tate refined this by describing

[1] Of course, a supersingular elliptic curve isn't singular. A purist perhaps would like to say "an elliptic curve with supersingular j-value". However we will adopt the generally used terminology "supersingular elliptic curve" instead.

[2] The prime number ℓ is assumed to be invertible in the base field.

the structure of the endomorphism algebra $\mathrm{End}^0(A)$ (working in the isogeny category over κ) in terms of the Weil q-integer of A, with $q = \#\kappa$; see [**121**, Thm. 1]. It follows from Tate's work (see 1.6.2.5) that an abelian variety A over a finite field κ admits sufficiently many complex multiplications in the sense that its endomorphism algebra $\mathrm{End}^0(A)$ contains a CM subalgebra[3] L of rank $2\dim(A)$. We will call such an abelian variety (in any characteristic) a CM *abelian variety* and the embedding $L \to \mathrm{End}^0(A)$ a CM *structure* on A.

Grothendieck showed that over any algebraically closed field K, an abelian variety that admits sufficiently many complex multiplications is isogenous to an abelian variety defined over a finite extension of the prime field [**89**]. This was previously known in characteristic zero (by Shimura and Taniyama), and in that case there is a number field $K' \subset K$ such that the abelian variety can be defined over K' (in the sense of 1.7.1). However in positive characteristic such abelian varieties can fail to be defined over a finite subfield of K; examples exist in every dimension > 1 (see Example 1.7.1.2).

Abelian varieties in mixed characteristic. In characteristic zero, an abelian variety A gives a representation of the endomorphism algebra $D = \mathrm{End}^0(A)$ on the Lie algebra $\mathrm{Lie}(A)$ of A. If A has complex multiplication by a CM algebra L of degree $2\dim(A)$ then the isomorphism class of the representation of L on $\mathrm{Lie}(A)$ is called the CM *type* of the CM structure $L \hookrightarrow \mathrm{End}^0(A)$ on A (see Lemma 1.5.2 and Definition 1.5.2.1).

As we noted above, every abelian variety over a finite field is a CM abelian variety. Thus, it is natural to ask whether every abelian variety over a finite field can be "CM lifted" to characteristic zero (in various senses that are made precise in 1.8.5). One of the obstacles[4] in this question is that in characteristic zero there is the notion of CM type that is invariant under isogenies, whereas in positive characteristic whatever can be defined in an analogous way is *not* invariant under isogenies. For this reason we will use the terminology "CM type" only in characteristic zero.

For instance, the action of the endomorphism ring $R = \mathrm{End}(A_0)$ of an abelian variety A_0 on the Lie algebra of A_0 in characteristic $p > 0$ defines a representation of R/pR on $\mathrm{Lie}(A_0)$. Given an isogeny $f : A_0 \to B_0$ we get an identification $\mathrm{End}^0(A_0) = \mathrm{End}^0(B_0)$ of endomorphism algebras, but even if $\mathrm{End}(A_0) = \mathrm{End}(B_0)$ under this identification, the representations of this common endomorphism ring on $\mathrm{Lie}(A_0)$ and $\mathrm{Lie}(B_0)$ may well be non-isomorphic since $\mathrm{Lie}(f)$ may not be an isomorphism. Moreover, if we have a lifting A of A_0 over a local domain of characteristic 0, in general the inclusion $\mathrm{End}(A) \subset \mathrm{End}(A_0)$ is not an equality. If the inclusion $\mathrm{End}^0(A) \subset \mathrm{End}^0(A_0)$ is an equality then the character of the representation of $\mathrm{End}(A_0)$ on $\mathrm{Lie}(A_0)$ is the reduction of the character of the representation of $\mathrm{End}(A)$ on $\mathrm{Lie}(A)$. This relation can be viewed as an obstruction to the existence of CM lifting with the full ring of integers of a CM algebra operating on the lift; see 4.1.2, especially 4.1.2.3–4.1.2.4, for an illustration.

In the case when $\mathrm{End}(A_0)$ contains the ring of integers $\mathcal{O}_L$ of a CM algebra $L \subset \mathrm{End}^0(A_0)$ with $[L : \mathbb{Q}] = 2\dim(A_0)$, the representation of $\mathcal{O}_L/p\mathcal{O}_L$ on $\mathrm{Lie}(A_0)$ turns out to be quite useful, despite the fact that it is not an isogeny invariant. Its class in a suitable K-group will be called the *Lie type* of $(A_0, \mathcal{O}_L \hookrightarrow \mathrm{End}(A_0))$.

[3]A CM algebra is a finite product of CM fields; see Definition 1.3.3.1.

[4]surely also part of the attraction

The above discrepancy between the theories in characteristic zero and characteristic $p > 0$ is the basic phenomenon underlying this entire book. Before discussing its content, we recall the following theorem of Honda and Tate ([**50**, §2, Thm. 1] and [**121**, Thm. 2]).

> *For an abelian variety A_0 over a finite field κ there is a finite extension κ' of κ and an isogeny $(A_0)_{\kappa'} \to B_0$ such that B_0 admits a* CM *lifting over a local domain of characteristic zero with residue field κ'.*

This result has been used in the study of Shimura varieties, for settings where the ground field is an algebraic closure of $\mathbb{F}_p$ and isogeny classes (of structured abelian varieties) are the objects of interest; see [**135**]. Our starting point comes from the following questions which focus on controlling ground field extensions and isogenies.

> *For an abelian variety A_0 over a finite field κ, to ensure the existence of a* CM *lifting over a local domain with characteristic zero and residue field κ' of finite degree over κ,*
> (a) *may we choose $\kappa' = \kappa$?*
> (b) *is an isogeny $(A_0)_{\kappa'} \to B_0$ necessary?*

These questions are formulated in various precise forms in 1.8.

An isogeny is necessary. Question (b) was answered in 1992 (see [**93**]) as follows.

> *There exist (many) abelian varieties over $\overline{\mathbb{F}}_p$ that do not admit any* CM *lifting to characteristic zero.*

The main point of [**93**] is that a CM liftable abelian variety over $\overline{\mathbb{F}}_p$ can be defined over a small finite field. This idea is further pursued in Chapter 3, where the *size*, or more accurately the *minima*[5] of the size, of all possible *fields of definition* of the p-divisible group of a given abelian variety over $\overline{\mathbb{F}}_p$ is turned into an obstruction for the existence of a CM lifting to characteristic 0. This is used to show (in 3.8.3) that in "most" isogeny classes of non-ordinary abelian varieties of dimension $\geqslant 2$ over finite fields there is a member that has no CM lift to characteristic 0. (In dimension 1 a CM lift to characteristic 0 always exists, over the valuation ring of the minimal possible p-adic field, by Deuring Lifting Theorem; see 1.7.4.6.) We also provide effectively computable examples of abelian varieties over explicit finite fields such that there is no CM lift to characteristic 0.

A field extension might be necessary—depending on what you ask. Bearing in mind the necessity to modify a given abelian variety over a finite field to guarantee the existence of a CM lifting, we rephrase question (a) in a more precise version (a)′ below.

> (a)′ *Given an abelian variety A_0 over a finite field κ of characteristic p, is it necessary to extend scalars to a strictly larger finite field $\kappa' \supset \kappa$ (depending on A_0) to ensure the existence of a κ'-rational isogeny $(A_0)_{\kappa'} \to B_0$ such that B_0 admits a* CM *lifting over a characteristic* 0 *local domain R with residue field κ'?*

It turns out there are two quite different answers to question (a)′, depending on whether one requires the local domain R of characteristic 0 to be normal. The subtle distinction between using normal or general local domains for the lifting

[5] The size of a finite field κ_1 is *smaller* than the size of a finite field κ_2 if κ_1 is isomorphic to a subfield of κ_2, or equivalently if $\#\kappa_1 \mid \#\kappa_2$. Among the sizes of a family of finite fields there may not be a unique minimal element.

went unnoticed for a long time. Once this distinction came in focus, answers to the resulting questions became available.

If we ask for a CM lifting over a *normal* domain up to isogeny, in general a base field extension *before* modification by an isogeny is necessary. This is explained in 2.1.2, where we formulate the "residual reflex obstruction", the idea for which goes as follows. Over an algebraically closed field K of characteristic zero, we know that a simple CM abelian variety B with K-valued CM type Φ (for the action of a CM field L) is defined over a number field in K containing the reflex field $E(\Phi)$ of Φ. Suppose that for every K-valued CM type Φ of L, the residue field of $E(\Phi)$ at any prime above p is not contained in the finite field κ with which we began in question (a). In such cases, for every CM structure $L \to \mathrm{End}^0(A_0)$ on A_0 and any abelian variety B_0 over κ which is κ-isogenous to A_0, there is no L-linear CM lifting of B_0 over a *normal* local domain R of characteristic zero with residue field κ.[6] In 2.3.1–2.3.3 we give such an example, a supersingular abelian surface A_0 over $\mathbb{F}_{p^2}$ with $\mathrm{End}(A_0) = \mathbb{Z}[\zeta_5]$ for any $p \equiv \pm 2 \pmod 5$. A much broader class of examples is given in 2.3.5, consisting of absolutely simple abelian varieties (with arbitrarily large dimension) over $\mathbb{F}_p$ for infinitely many p.

Note that passing to the normalization of a complete local noetherian domain generally enlarges the residue field. Hence, if we drop the condition that the mixed characteristic local domain R be normal then the obstruction in the preceding consideration dissolves. And in fact we were put on the right track by mathematics itself. The phenomenon is best illustrated in the example in 4.1.2, which is the same as the example in 2.3.1 already mentioned: an abelian surface C_0 over $\mathbb{F}_{p^2}$ with CM order $\mathbb{Z}[\zeta_5]$ that, even up to isogeny, is not CM liftable to a *normal* local domain of characteristic zero. On the other hand, this abelian surface C_0 is CM liftable to an abelian scheme C over a mixed characteristic non-normal local domain of characteristic zero, though the maximal subring of $\mathbb{Z}[\zeta_5]$ whose action lifts to C is non-Dedekind locally at p; see 4.1.2.[7] This example is easy to construct, and the proof of the existence of a CM lifting, possibly after applying an $\mathbb{F}_{p^2}$-rational isogeny, is not difficult either.

In Chapter 4 we show that the general question of existence of a CM lifting after an appropriate isogeny can be reduced to the same question for (a mild generalization of) the example in 4.1.2, enabling us to prove:

> *every abelian variety A_0 over a finite field κ admits an isogeny $A_0 \to B_0$ over κ such that B_0 admits a* CM *lifting to a mixed characteristic local domain with residue field κ.*

There are refined lifting problems, such as specifying at the beginning which CM structure on A_0 is to be lifted, or even what its CM type should be on a geometric fiber in characteristic 0. These matters will also be addressed.

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[6]The source of obstructions is that the *base field* κ might be too small to contain at least one characteristic p residue field of the reflex field $E(\Phi)$ for at least one CM type Φ on L. Thus, the field of definition of the generic fiber of the hypothetical lift may be too big. Likewise, an obstruction for question (b) is that the field of definition of the p-divisible group $A_0[p^\infty]$ may be too big (in a sense that is made precise in 3.8.3 and illustrated in 3.8.4–3.8.5).

[7]No modification by isogeny is necessary in this example, but the universal deformation for C_0 with its $\mathbb{Z}[\zeta_5]$-action is a *non-algebraizable* formal abelian scheme over $W(\mathbb{F}_{p^2})$.

Our basic method is to "localize" various CM lifting problems to the corresponding problems for p-divisible groups. Although global properties of abelian varieties are often lost in this localization process, the non-rigid nature of p-divisible groups can be an advantage. In Chapter 3 the size of fields of definition of a p-divisible group in characteristic p appears as an obstruction to the existence of CM lifting. The reduction steps in Chapter 4 rely on a classification and descent of CM p-divisible groups in characteristic p with the help of their Lie types (see 4.2.2, 4.4.2). In addition, the "Serre tensor construction" is applied to p-divisible groups, both in characteristic p and in mixed characteristic $(0, p)$; see 1.7.4 and 4.3.1 for this general construction.

Survey of the contents. In **Chapter 1** we start with a survey of general facts about CM abelian varieties and their endomorphism algebras. In particular, we discuss the deformation theory of abelian varieties and p-divisible groups, and we review results in Honda-Tate theory that describe isogeny classes and endomorphism algebras of abelian varieties over a finite field in terms of Weil integers. We conclude by formulating various CM lifting questions in 1.8. These are studied in the following chapters. We will see that the questions can be answered with some precision.

In **Chapter 2** we formulate and study the "residual reflex condition". Using this condition we construct several examples of abelian varieties over finite fields κ such that, even after applying a κ-isogeny, there is no CM lifting to a *normal local domain* with characteristic zero and residue field of finite degree over κ; see 2.3. It is remarkable that many such examples exist, but we do not know whether we have characterized all possible examples; see 2.3.7.

We then study algebraic Hecke characters and review part of the theory of complex multiplication due to Shimura and Taniyama. Using the relationship between algebraic Hecke characters for a CM field L and CM abelian varieties with CM by L (the precise statement of which we review and prove), we use global methods to show that the residual reflex condition is the *only* obstruction to the existence of CM lifting up to isogeny over a *normal* local domain of characteristic zero. We also give another proof by local methods (such as p-adic Hodge theory).

In **Chapter 3** we take up methods described in [**93**]. In that paper classical CM theory in characteristic zero was used. Here we use p-divisible groups instead of abelian varieties and show that the size of fields of definition of a p-divisible group in characteristic p is a non-trivial obstruction to the existence of a CM lifting. In 3.3 we study the notion of isogeny for p-divisible groups over a base scheme (including its relation with duality). We show, in one case of the CM lifting problem left open in [**93**, Question C], that an isogeny is necessary. Our methods also provide effectively computed examples. Some facts about CM p-divisible groups explained in 3.7 are used in 3.8 to get an upper bound of a field of definition for the closed fiber of a CM p-divisible group.

In Appendix 3.9, we use the construction (in 3.7) of a p-divisible group with any given p-adic CM type over the reflex field to produce a semisimple abelian crystalline p-adic representation of the local Galois group such that its restriction to the inertia group is "algebraic" with algebraic part that we may prescribe arbitrarily in accordance with some necessary conditions (see 3.9.4 and 3.9.8).

In **Chapter 4** we show CM liftability after an isogeny over the finite ground field (lifting over a characteristic zero local domain that need not be normal). That is,

> *every* CM *structure* $(A_0, L \to \mathrm{End}^0(A_0))$ *over a finite field* κ *has an isogeny over* κ *to a* CM *structure* $(B_0, L \to \mathrm{End}^0(B_0))$ *that admits a* CM *lifting;*

(see 4.1.1). This statement is immediately reduced to the case when L is a CM field (not just a CM algebra) and the whole ring $\mathcal{O}_L$ of integers of L operates on A_0, which we assume.

Our motivation comes from the proof in 4.1.2 (using an algebraization argument at the end of 4.1.3) that the counterexample in 2.3.1 to CM lifting over a *normal* local domain satisfies this property. In general, after an easy reduction to the isotypic case, we apply the Serre-Tate deformation theorem to localize the problem at p-adic places v of the maximal totally real subfield L^+ of a CM field $L \subseteq \mathrm{End}^0(A_0)$ of degree $2\dim(A_0)$. This reduces the existence of a CM lifting for the abelian variety A_0 to a corresponding problem for the CM p-divisible group $A_0[v^\infty]$ attached to v.[8]

We formulate several properties of v with respect to the CM field L; any one of them ensures the existence of a CM lifting of $A_0[v^\infty]_{\overline{\kappa}}$ after applying a κ-isogeny to $A_0[v^\infty]$ (see 4.1.6, 4.1.7, and 4.5.7). These properties involve the ramification and residue fields of L and L^+ relative to v. If v violates all of these properties then we call it *bad* (with respect to L/L^+ and κ). Let $L_v := L \otimes_{L^+} L^+_v$. After applying a preliminary κ-isogeny to arrange that $\mathcal{O}_L \subset \mathrm{End}(A_0)$, for v that are not bad we apply an $\mathcal{O}_L$-linear κ-isogeny to arrange that the Lie type of the $\mathcal{O}_{L,v}$-factor of $\mathrm{Lie}(A_0)$ (i.e., its class in a certain K-group of $(\mathcal{O}_{L,v}/(p)) \otimes \kappa$-modules) is "self-dual". Under the self-duality condition (defined in 4.4.3) we produce an $\mathcal{O}_{L,v}$-linear CM lifting of $A_0[v^\infty]_{\overline{\kappa}}$ by specializing a suitable $\mathcal{O}_{L,v}$-linear CM v-divisible group in mixed characteristic; see 4.4.6. We use an argument with deformation rings to eliminate the intervention of $\overline{\kappa}$: if every p-adic place v of L^+ is not bad then there exists a κ-isogeny $A_0 \to B_0$ such that $\mathcal{O}_L \subset \mathrm{End}(B_0)$ and the pair $(B_0, \mathcal{O}_L \hookrightarrow \mathrm{End}(B_0))$ admits a lift to characteristic 0 without increasing κ.

If some p-adic place v of the totally real field L^+ is bad then the above argument does not work because in that case no member of the $\mathcal{O}_{L,v}$-linear κ-isogeny class of the p-divisible group $A_0[v^\infty]$ has a self-dual Lie type. Instead we change $A_0[v^\infty]$ by a suitable $\mathcal{O}_{L,v}$-linear κ-isogeny so that its Lie type becomes as symmetric as possible, a condition whose precise formulation is called "striped". Such a p-divisible group is shown to be isomorphic to the Serre tensor construction applied to a special class of 2-dimensional p-divisible groups of height 4 that are similar to the ones arising from the abelian surface counterexamples in 2.3.1; we call these *toy models* (see 4.1.3, especially 4.1.3.2).

These "toy models" are sufficiently special that we can analyze their CM lifting properties directly; see 4.2.10 and 4.5.15(iii). After this key step we deduce the existence of a CM lifting of $A_0[v^\infty]_{\overline{\kappa}}$ from corresponding statements for (the p-divisible group version of) toy models. In the final step, once again we use deformation theory to produce an abelian variety B_0 isogenous to (the original) A_0 over κ and a CM lifting of B_0 over a possibly *non-normal* 1-dimensional complete local noetherian domain of characteristic 0 with residue field κ. Although $\mathcal{O}_L$ acts

[8]See 1.4.5.3 for the statement of the Serre–Tate deformation theorem, and 2.2.3 and 4.6.3.1 for a precise statement of the algebraization criterion that is used in this localization step.