Problem Books in Mathematics

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A Problem Seminar

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PREFACE

There was once a bumper sticker that read, "Remember the good old days when *air* was clean and *sex* was dirty?" Indeed, some of us are old enough to remember not only *those* good old days, but even the days when Math was *fun*(!), not the ponderous THEOREM, PROOF, THEOREM, PROOF,..., but the whimsical, "I've got a good problem."

Why did the mood change? What misguided educational philosophy transformed graduate mathematics from a passionate activity to a form of passive scholarship?

In less sentimental terms, why have the graduate schools dropped the Problem Seminar? We therefore offer "A Problem Seminar" to those students who haven't enjoyed the fun and games of problem solving.

CONTENTS

FORMAT

This book has three parts: first, the list of problems, briefly punctuated by some descriptive pages; second, a list of hints, which are merely meant as words to the (very) wise; and third, the (almost) complete solutions. Thus, the problems can be viewed on any of three levels: as somewhat difficult challenges (without the hints), as more routine problems (with the hints), or as a textbook on "how to solve it" (when the solutions are read). Of course it is our hope that the book can be enjoyed on any of these three levels.

PROBLEMS

1. Derive the operations $+$, $-$, $\times$, and $\div$ from $-$ and reciprocal.

2. Invent a single (binary) operation from which $+$, $-$, $\times$, and $\div$ can be derived.

3. The multiplication of two complex numbers $(a+bi)\cdot(x+yi) = ax - by + (bx + ay)i$ appears to need 4 real multiplications $(a \cdot x, b \cdot y, b \cdot x, a \cdot y)$, but does it really? If additions are free, can this same job be accomplished in 3 real multiplications? In 2?

4. A microbe either splits into two perfect copies of itself or else disintegrates. If the probability of splitting is p, what is the probability that one microbe will produce an everlasting colony?

5. Given any n distinct points in the plane, show that one of the angles determined by them is $\leqslant \pi/n$ (the 0 angle counts).

6. Prove that every sequence (of real numbers) contains a monotone subsequence.

7. Suppose $f(x) = x + x^2/n$, and form $\underbrace{f(f(f\ldots f(x))\ldots)}_{n}$ (same n, n a positive integer). What is the limiting behavior as $n \to \infty$?

8. Devise an experiment which uses only tosses of a fair coin, but which has success probability $\frac{1}{3}$. Do the same for any success probability p, $0 \leqslant p \leqslant 1$.

9. We alternate writing down binary digits after a decimal point, thereby producing a real number in the interval $[0, 1]$. You win if this number is transcendental. Can you force a win?

10. At a certain corner, the traffic light is green for 30 seconds and then red for 30 seconds. On the average, how much time is lost at this corner?

11. Prove that there is no equilateral triangle all of whose vertices are plane lattice points. (How about three-dimensional lattice points?)

12. Prove that a sequence of positive numbers, each of which is less than the average of the previous two, is convergent.

13. $x_{n+1} = \frac{1}{2}(x_n + 1/x_n)$, x_0 a given *complex* number. Discuss convergence.

14. $x_{n+1} = (x_n + x_{n-1})/2$, x_0, x_1 given. Express $\lim x_n$ explicitly.

15. If a set of positive integers has sum n, what is the biggest its product can be?

16. Given a convergent series of positive terms, Σa_n, prove that $\Sigma \sqrt[n]{a_1 a_2 \ldots a_n}$ must also be convergent.

17. What is the lowest degree monic polynomial which vanishes identically on the integers (mod 100)? (And generally (mod n)?)

18. Evaluate $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$.

19. Prove that, at any party, two people have the same number of friends present.

20. If n is any integer greater than 1, then n does *not* divide $2^n - 1$.

21. Prove that every non-multiple of 3 is congruent to a power of 2 (mod 3^n).

22. How many perfect squares are there (mod 2^n)?

23. Maximize $2^{-x} + 2^{-1/x}$ over $(0, \infty)$.

24. N distinct non-collinear points are given. Prove that they determine at least N distinct lines.

25. Given that $f(x)$ increases from 0 to 1 as x does, prove that the graph of $y = f(x)$ $(0 \leqslant x \leqslant 1)$ can be covered by n rectangles with sides parallel to the axes and each having area $1/n^2$.

26. Given a finite collection of closed squares of total area 3, prove that they can be arranged to cover the unit square.

27. Given a finite collection of squares of total area $\frac{1}{2}$, show that they can be arranged so as to fit in a unit square (with no overlaps).

28. Devise the smallest plane set such that no point is at a rational distance from all points of the set.

29. Given an infinite number of points in the plane with all the mutual distances integers, prove that the points are all collinear.

30. If a, b are positive integers, then $(a+\frac{1}{2})^n + (b+\frac{1}{2})^n$ is an integer only for finitely many positive integers, n.

31. Given that $f(x, y)$ is a polynomial in x for each fixed y, and $f(x, y)$ is a polynomial in y for each fixed x, must $f(x, y)$ be a polynomial in x and y?

32. Prove that the product of 3 consecutive integers is never a perfect power (i.e., a perfect square or a perfect cube, etc.).

33. Given a region whose boundary is a simple polygon, prove that it contains a disc with radius larger than area/perimeter.

34. I choose an integer from 0 through 15. You ask me 7 yes or no questions. I answer them all, but I am allowed to *lie* once. (I needn't, but I am allowed to.) Determine my number!

35. Given any bounded plane region, prove that there are three *concurrent* lines that cut it into six pieces of equal area.

36. Given any bounded plane region, prove that there is a point through which no line trisects the area.

37. $a, b, c, d, \dots$ are positive numbers. Prove

$$\sqrt{a+b+c+d+\cdots} + \sqrt{b+c+d+\cdots} + \sqrt{c+d+\cdots} + \cdots \geqslant \sqrt{a+4b+9c+16d+\cdots}\,.$$

38. Show that the number 16 is a perfect *8th* power (mod p) for any prime p.

39. The points of the plane are each colored either red, yellow, or blue. Prove that there are two points of the same color having mutual distance 1.

40. Assume that the points of the plane are each colored red or blue. Prove that one of these colors contains pairs of points at *every* mutual distance.

41. Given a simple plane arc of length more than 1, prove that for some n there are more than n points on the arc whose mutual distances are all at least $1/n$.

42. Good coins weigh 10 gm, bad ones 9 gm. Given 4 coins and a scale (not a balance, but a true scale), determine which are which in only 3 weighings.

43. Let $[\alpha, \beta]$ be an interval which contains no integers. Show that there is a positive integer n such that $[n\alpha, n\beta]$ still contains no integers but has length at least $\frac{1}{6}$.

44. Prove that the integers $[(\sqrt{2}+1)^n]$ are alternately even and odd.

45. Prove that $\min_k (k + [n/k]) = [\sqrt{4n+1}\,]$. (Here n is a given positive integer and k varies over all positive integers.)

46. Let α, β be positive irrationals. Show that the sets $[n\alpha]$ and $[n\beta]$, $n = 1, 2, 3, \ldots$, are complements iff $1/\alpha + 1/\beta = 1$.

47. Suppose we "sieve" the integers as follows: we choose $a_1 = 1$ and then delete $a_1 + 1 = 2$. The next term is 3, which we call a_2, and then we delete $a_2 + 2 = 5$. Thus, the next available integer is $4 = a_3$, and we delete $a_3 + 3 = 7$, etc. Thereby we leave the integers $1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, \ldots$. Find a formula for a_n.

48. Call an integer square-full if each of its prime factors occurs to the second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

49. We term two sets "almost disjoint" if their intersection is finite. What is the largest (cardinality) collection of sets of integers which are pairwise almost disjoint?

50. Split a beer *three ways*. To split a beer *two ways* you let the first man divide it into what he thinks are two equal parts and then let the second man choose one of them. Both are then satisfied. How can three do this?

51. Define x_n by $x_n = x_{n-1} + \frac{1}{2}x_{n-2}$, $x_0 = 0$, $x_1 = 1$. Prove that for $n > 8$, x_n is not an integer.

52. Let $a_1, a_2, \ldots, a_k$ be positive integers and let S be the set of all positive integers not divisible by any of them. Prove that the density of S is at least $(1 - 1/a_1)(1 - 1/a_2)\ldots(1 - 1/a_k)$.

53. Given any n real pairs (x_i, y_i), with the x_i all distinct, prove that the interpolation problem $P(x_i) = y_i$, $i = 1, 2, \ldots, n$, can be solved by a polynomial P, all of whose zeros are real.

54. Is there a non-trivial function $f(x)$, continuous on the whole line, which satisfies the functional equation $f(x) + f(2x) + f(3x) \equiv 0$?

55. Give examples of:
   (1) An infinite group with no infinite *proper* subgroup.
   (2) A field isomorphic to a proper subfield.
   (3) A ring with no maximal ideals.

56. $n_1, n_2, n_3, \ldots$ is a sequence of positive integers with the property $n_{k+1} > n_{n_k}$. Show that it must be the sequence $1, 2, 3, 4, \ldots$.

57. (A CHEERFUL FACT ABOUT...) Given a right triangle and a finite set of points inside it, prove that these points can be

connected by a path of line segments the sum of whose squares is bounded by *the square of the hypotenuse*.

58. Batter A has a higher batting average than batter B for the first half of the season *and* A also has a higher batting average than B for the second half of the season. Does it follow that A has a better batting average than B for the whole season?

Estimation Theory

This whole topic is perhaps born out of the shortage of exact formulas. In many (most?) situations a quantity is sought which cannot be expressed in simple closed form, but can only be estimated. Of course the game is to estimate it *well*, but how? That is, how do you know that you've done a good job?

The answer is that this game is played on two "fronts." One estimates above (finds something definitely larger than the desired quantity), and one also estimates below (finds something definitely smaller). In many cases the arguments leading to these two bounds may be totally unrelated and quite ad hoc. Each argument *alone* gives no assurance that it is getting anywhere near the truth— *BUT*, when the two arguments give results which are close to one another, they PROVE each other out. It is then that one knows he has been *clever* and accurate!

There is a nice freedom in this subject. Basically there are no *wrong* answers, just better or worse ones depending on how close the resultant upper and lower bounds turn out to be.

59. If $e^{e^x} = \sum a_n x^n$, estimate a_n.

60. If $x_0 = 1$, $x_{n+1} = x_n + 1/x_n$, then $x_n \to \infty$ (why?). Estimate how fast.

61. $\sum_{n=0}^{\infty} x^{n^2}$ tends to ∞ as $x \to 1^-$. How fast?

62. The equation $x^n + x = 1$ has a unique positive solution $x(n)$, which approaches 1 as $n \to \infty$. Estimate how fast.

63. $\underbrace{\sin \sin \sin \ldots \sin}_{n}(\pi/2)$ goes to 0 as $n \to \infty$. Estimate how fast.

64. The function $f(x) = \sum_{n=0}^{\infty} 1/(2^n + x)$ goes to 0 as $x \to \infty$. How fast?

65. Estimate the largest collection of triples one can choose from n elements such that no two of them overlap in more than one element.

66. Consider the sequence $1, 2, 3, 4, 6, 8, 9, 12, 16, 18, \ldots$ of positive integers composed of 2's and 3's (that is of the form $2^a 3^b$, a, b non-negative integers) and arranged in increasing order. Prove that the ratio of successive terms approaches 1.

67. Find, asymptotically, how many lattice points in the square $0 < x \leqslant N$, $0 < y \leqslant N$, are "visible" from the origin. (The point $(5, 8)$ is visible from the origin since no other lattice point blocks the view, whereas the point $(6, 8)$ is blocked by the point $(3, 4)$, i.e., the line from the origin to $(6, 8)$ hits $(3, 4)$.)