Hilbert Space and Quantum Mechanics

Franco Gallone

World Scientific

Hilbert Space and Quantum Mechanics

Franco Gallone

Università degli Studi di Milano, Italy

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

HILBERT SPACE AND QUANTUM MECHANICS

ISBN 978-981-4635-83-7

In-house Editor: Ng Kah Fee

Printed in Singapore

Hilbert Space and Quantum Mechanics

To Kissy, Lilith, Malcy, Micio,
who taught me how to stay focused

Preface

The subjects of this book are the mathematical foundations of non-relativistic quantum mechanics and the mathematical theory they require. In its mathematical part, this book aims at expounding in a complete and self-contained way the mathematical basis for "mathematical" quantum mechanics, namely the branch of mathematical physics that was constructed by David Hilbert, John von Neumann and other mathematicians, notably George Mackey, in order to systematize quantum mechanics, and which was presented in book form for the first time by von Neumann in 1932 (Neumann, 1932). In von Neumann's approach, the language of quantum mechanics is the theory of linear operators in Hilbert space.

Von Neumann's book was the result of work which had been done previously over several years. Hilbert, who had been consulted on numerous aspects of quantum mechanics since its inception, began in 1926 a systematic study of its mathematical foundations. Hilbert taught the course "Mathematical Methods of Quantum Theory" in the academic year 1926-27, and a summary of Hilbert's lessons was published in the spring of 1927 by Hilbert himself and his assistants Lothar Nordheim and von Neumann (Hilbert *et al.*, 1927). In their view, the mathematical framework suitable for quantum mechanics was the mathematical structure that was defined in an abstract way and called a Hilbert space by von Neumann in 1927. Furthermore, between 1926 and 1932, von Neumann proved a number of theorems about operators in Hilbert space which bore upon quantum mechanics (among them, the spectral theorem for unbounded self-adjoint operators), and so did the mathematicians Marshall Stone and Hermann Weyl, who had a keen interest in quantum mechanics. Thus, the theory of linear operators in Hilbert space was actually born as the mathematical basis for quantum mechanics.

Quantum mechanics and the theory of Hilbert space operators constitute one of those rare examples in which there is complete correspondence between physical and mathematical concepts (another example is Euclidean geometry). Actually, it is one of the most stunning examples of "the unreasonable effectiveness of mathematics in the natural sciences" (E.P. Wigner). Unfortunately, this aspect of quantum mechanics is almost completely overlooked in most quantum mechanics textbooks, where too many subtle points are dealt with by means of mathematical shortcuts which not only can hardly convince a mathematically aware reader but also blot out physical subtleties. The main reason for this is that, in the community of physicists, Dirac's quantum mechanics (Dirac, 1958, 1947, 1935, 1930) is by far more popular than von Neumann's quantum mechanics, perhaps exactly because the former requires almost no mathematics. For instance, the idea that self-adjoint operators have a critical domain is almost completely missing in standard quantum mechanics textbooks; however, the domain of an unbounded self-adjoint operator represents exactly the pure states in which the fundamental statistical quantities (expected result and uncertainty) are defined for the observable represented by that operator. This point gets hopelessly blurred in most quantum mechanics books, which treat unbounded observables — like energy, position, momentum, orbital angular momentum — as if they were represented by self-adjoint operators defined on the entire space, while this is impossible on account of the Hellinger–Toeplitz theorem. Another example is the relation existing between the physical idea of compatibility of two observables and the mathematical idea of commutativity of the operators that represent them; for self-adjoint operators, the right notion of commutativity is subtler than the one usually found in quantum mechanics books and depends on the representations of the operators as projection valued measures; however it is exactly through this subtler notion that the physical essence of compatibility can be really grasped. More than anything else, the real way to understand why quantum observables are represented by self-adjoint operators is through the spectral theorem, since quantum observables arise most naturally as projection valued measures, but this is usually outside the scope of standard quantum mechanics books.

One last word about the mathematical framework for quantum mechanics presented in this book. It is undoubtedly very interesting and useful to treat quantum mechanics in the framework of mathematical structures more general than Hilbert space theory, especially in order to study quantum mechanics of systems with an infinite number of degrees of freedom. However, quantum mechanics in Hilbert space is an enthralling subject in its own right, mainly because it is here that one can see most clearly how the mathematical structure is linked to the physical theory in an almost necessary way.

Most books about fundamental quantum mechanics use results in the theory of Hilbert space operators without proving them, while most books about Hilbert space operators do not treat quantum mechanics; moreover, they often use fairly advanced results from other branches of mathematics assuming the reader to be already familiar with them, but this is seldom true. The aim of this book is not to be a complete treatise about Hilbert space operators, but to give a really self-contained treatment of all the elements of this subject that are necessary for a sound and mathematically accurate exposition of the principles of quantum mechanics; this exposition is the object of the final chapters of the book. The main characteristic of the book is that the mathematical theory is developed only assuming familiarity with elementary analysis. Moreover, all the proofs in the book are carried out in a very detailed way. These features make the book easily accessible to readers with only the mathematical experience offered by undergraduate education in mathematics or in physics, and also ideal for individual study. The principles of quantum mechanics are discussed with complete mathematical accuracy and an effort is always made to trace them back to the experimental reality that lies at their root. The treatment of quantum mechanics is axiomatic, with definitions followed by propositions proved in a mathematical fashion. No previous knowledge of quantum mechanics is required. The level of this book is intermediate between advanced undergraduate and graduate. It is a purely theoretical book, in which no exercises are provided.

After the first chapter, whose function is mainly to fix notation and terminology, the first part of the book (Chapters 2–9) is devoted to an exposition of the elements of real and abstract analysis that are needed later in the study of operators in Hilbert space. The reason for this is to make it really self-contained and avoid proving theorems by means of other fairly advanced theorems outside this book. In particular, the chapter devoted to metric spaces (Chapter 2) contains results which are not completely elementary but are necessary in order to prove (in Chapter 6) the theorem about Borel functions that plays an essential role in proving the spectral theorems (in Chapter 15). The chapters about measure and integration (Chapters 5–9) contain results about extensions of measures which are not to be found in first level books on measure theory but which are essential in order to study commuting self-adjoint operators, and also the Riesz–Markov theorem about positive linear functionals which plays an essential role in proving the spectral theorems. Actually, Chapters 1–2 and 5–9 could by themselves be a short book about measure and integration. Chapters 3 and 4 deal with that part of the theory of linear operators in normed spaces that is used later in the study of Hilbert space operators. Moreover, the Stone–Weierstrass approximation theorem is proved in Chapter 4; this theorem plays an essential role in proving the spectral theorems.

The second part of this book (Chapters 10–18) is its core, and contains a treatment of the theory of linear operators in Hilbert space which is particularly well suited for the discussion of the mathematical foundations of quantum mechanics presented later in the book. It contains the spectral theorems for unitary and for self-adjoint operators, one-parameter unitary groups and Stone's theorem, theorems about commuting operators and invariant subspaces, trace class operators, and also Wigner's theorem and the real line special case of Bargmann's theorem about automorphisms of projective Hilbert spaces.

The theory of Hilbert space operators is the backbone of the third and final part of the book, which consists of two chapters (19 and 20). The first of these is by far the longest chapter in the book and endeavours to present the principles of non-relativistic quantum mechanics in a mathematically accurate way, with also an unstinting effort to present some possible physical reasoning behind the constructs that are considered. Since the predictions provided by quantum mechanics are in general statistical ones, in the first part of this chapter general statistical ideas are introduced and it is examined how these ideas are implemented in classical theories; later in the chapter, the statistical aspects of quantum mechanics are compared and contrasted with the same aspects of classical theories. The final chapter deals with an important example of how quantum observables can arise in connection with symmetry principles; moreover, it presents the Stone–von Neumann uniqueness theorem about canonical commutation relations.

Although the book's length might make it difficult to use it as a textbook for a single course, parts of it can easily be used in that way for various courses. Here are some concrete suggestions:

- Chapters 1, 2, 5, 6, 7, 8, 9 for a one-semester course in Real Analysis or in Measure Theory (intermediate, could be either undergraduate or graduate, mathematics);
- Chapters 3, 4, 10, 11, 12, 13, 14, 15, 16, 17, 18 for a two-semester course in Operators in Hilbert Space (graduate, mathematics and physics);
- Chapters 19, 20 (using without proof a large number of results from the previous chapters) for a one-semester course in Mathematical Foundations of Quantum Mechanics (graduate, mathematics and physics).

To make cross-reference as easy as possible, almost every bit of this book is marked with three numbers, the first for the chapter, the second for the section, and the third for the position within the section. Comments also are marked in this way, and they are called "remarks". As already mentioned, all the proofs in this book are written in minute detail; in them, however, previous results are always quoted simply by means of the three numbers code, without spelling them out. This should enable experts to pursue the logic of a proof without too many diversions, and beginners to receive all the support they might need.

I wish to thank Roberto Palazzi for the great job he did of preparing the LaTeX files for the book, and also for useful mathematical comments.

Franco Gallone

Contents

Chapter 1

Sets, Mappings, Groups

Most readers are likely to have a working familiarity with most of the subjects of this introductory chapter. For them, the main function of this chapter is to fix the notation and the terminology that will be used throughout this book and provide ready reference inside the book.

1.1 Symbols, sets, relations

The reader is assumed to be already familiar with the topics of this section, which is only intended for future reference.

1.1.1 *Sets of numbers*

Symbol	Meaning
$\mathbb{N}$	the set of all positive integers, i.e. $\{1, 2, 3, ...\}$
$\mathbb{Z}$	the set of all integers, i.e. $\{0, \pm 1, \pm 2, ...\}$
$\mathbb{Q}$	the set of all rational numbers, i.e. $\{m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$
$\mathbb{R}$	the set of all real numbers
$[0, \infty)$	the set of all non-negative real numbers
$(0, \infty)$	the set of all positive real numbers
$\mathbb{C}$	the set of all complex numbers

The complex field is always meant to be $\mathbb{R}^2$ endowed with the two operations:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$
$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1),$$

and $\mathbb{C}$ denotes the set $\mathbb{R}^2$ when $\mathbb{R}^2$ is endowed in this way.

For a complex number $z := (a_1, a_2)$, we define:

$$\operatorname{Re} z := a_1, \operatorname{Im} z := a_2, \overline{z} := (a_1, -a_2), |z| := \sqrt{a_1^2 + a_2^2}.$$

The subset $\{(a, 0) : a \in \mathbb{R}\}$ of $\mathbb{C}$ is identified with $\mathbb{R}$, identifying $(a, 0)$ with a. With this identification, for a complex number z we have $\overline{z}z = z\overline{z} = |z|^2$, and the absolute value of a real number a coincides with $|a|$. Identifying $a \in \mathbb{R}$ with $(a, 0)$ and defining $i := (0,1)$, we also have $(a_1, a_2) = a_1 + ia_2$. When for a complex number z we write $0 \leq z$ (or $0 < z$, $z \leq 0$, $z < 0$), we mean $\operatorname{Im} z = 0$ and $0 \leq \operatorname{Re} z$ (or $0 < \operatorname{Re} z$, $\operatorname{Re} z \leq 0$, $\operatorname{Re} z < 0$). More generally, outside the chapters devoted to measure and integration, when for a symbol x we write $0 \leq x$ or $x \geq 0$ we mean $x \in [0, \infty)$; similarly, by $0 < x$ or $x > 0$ we mean $x \in (0, \infty)$. However, in chapters from 5 to 9 by $0 \leq x$ or $x \geq 0$ we mean $x \in [0, \infty]$ and by $0 < x$ or $x > 0$ we mean $x \in (0, \infty]$ (i.e. we allow the case $x = \infty$; cf. 5.1.1).

It is always understood that the square root of a positive real number is taken to be positive.

1.1.2 *Proofs*

A proposition is a statement that is either true or false (but not both). By means of logical connectives and brackets, a new proposition can be defined starting from one or more given propositions. We assume known to the reader the logical connectives: "not", "and", "or" ("A or B" means "A or B or both"), "$\Rightarrow$" (if, then), "$\Leftrightarrow$" (if and only if).

Given two propositions P, Q, the proposition $P \Rightarrow Q$ is logically equivalent to the proposition $(\text{not}Q) \Rightarrow (\text{not}P)$, which is called the contrapositive form of $P \Rightarrow Q$. A proof that $(\text{not}Q) \Rightarrow (\text{not}P)$ is true, is called *proof by contraposition* of $P \Rightarrow Q$. The proposition $P \Rightarrow Q$ is also logically equivalent to the proposition $[P \text{ and } (\text{not}Q)] \Rightarrow [R \text{ and } (\text{not}R)]$, for any proposition R. A proof that there is a proposition R such that $[P \text{ and } (\text{not}Q)] \Rightarrow [R \text{ and } (\text{not}R)]$ is true, is called *proof by contradiction* of $P \Rightarrow Q$.

Suppose that, for each positive integer n, we are given a proposition P_n. From the principle of induction it follows that, if the propositions

(a) P_1,
(b) $P_n \Rightarrow P_{n+1}$ is true for each positive integer n

are true, then the proposition

(c) P_n is true for each positive integer n

is true. A proof that propositions a and b are true is called *proof by induction* of proposition c.

Often, for a proposition P, we will write "P" instead of "P is true" or "P holds". Propositions will be written in a rather informal style, mixing logical symbols and ordinary language.