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THE ELEMENTARY THEORY OF GROUPS

A GUIDE THROUGH THE PROOFS OF THE TARSKI CONJECTURES

EXPOSITIONS IN MATHEMATICS 60

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Authors
Prof. Dr. Benjamin Fine
Fairfield University
Department of Mathematics
1073 North Benson Road
Fairfield, CT 06430
USA
ben1902@aol.com

Prof. Dr. Anthony Gaglione
United States Naval Academy
Department of Mathematics
9E Mail Stop
Annapolis, MD 21401
USA
amg@usna.edu

Alexei Myasnikov
McGill University
Dept. of Mathematics and Statistics
805 Sherbrooke St. West
Montréal, QC H3A 2K6
Canada
alexeim@math.mcgill.ca

Prof. Dr. Gerhard Rosenberger
Heinrich-Barth-Str. 1
20146 Hamburg
Germany
rosenber@fim.uni-passau.de

Prof. Dr. Dennis Spellman
Temple University
Dept. of Mathematics
Broad and Montgomery
Philadelphia, PA 19122
USA

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Preface

In 1940, Alfred Tarski, the noted logician, asked three major questions about the elementary or first-order theory of the class of non-Abelian free groups. These were subsequently formalized into conjectures. The first of these *Tarski conjectures* about non Abelian free groups is that all non-Abelian free groups have exactly the same first-order theory. The second is that the natural embedding of one free group into another is an elementary embedding. This second conjecture implies the first. Finally Tarski asked if the elementary theory of the non-Abelian free groups is decidable, that is does there exist an algorithm to determine if a first order sentence is true or not within the class of non-Abelian free groups. The conjectures remained open for over fifty years.

In a series of papers from 1998–2006 the first two Tarski conjectures were answered in the affirmative by Olga Kharlampovich and Alexei Myasnikov [152, 153, 154, 155, 156] and independently by Zil Sela [233, 234, 235, 236, 237]. Kharlampovich and Myasnikov also proved Tarski conjecture 3. The proofs of both Kharlampovich and Myasnikov and of Sela involve long and complicated applications of algebraic geometry over free groups (Sela calls this Diophantine geometry) as well as an extension of methods to solve equations over free groups originally developed by Makanin and Razborov. The material necessary to understand these proofs is quite daunting even for accomplished group theorists and logicians. This book is an examination of the material on group theory and logic and on the general elementary theory of groups that is necessary to begin to understand the proofs. This material includes a complete exposition of the theory of fully residually free groups or limit groups as well a complete description of the algebraic geometry of free groups. Also included are introductory material on combinatorial and geometric group theory and first-order logic. There is then a short outline of the proof of the Tarski conjectures. We found that in many cases, group theorists don't know enough logic to understand the proof while the same is true for logicians, that is the logicians for the most part don't understand enough of the group theory. Part of our goal in this book is to correct this.

We first introduce some basic ideas and give some history.

The elementary theory of groups is tied to first-order logic and to model theory. We will look at elementary logic and model theory in more detail in Chapter 4. We start with a first-order language appropriate for group theory. This language, which we denote L_0, is the first-order language with equality containing a binary operation symbol $\cdot$ a unary operation symbol $^{-1}$ and a constant symbol 1. A *sentence* in L_0 is a logical expression using variables, the operational symbols above, equality, logical connectives and quantifiers. A *universal sentence* of L_0 is one of the form $\forall\overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ is a tuple of distinct variables, $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing at most the variables of $\overline{x}$. For example

$$\forall(x, y)\{xy = yx\}$$

is a universal sentence describing an Abelian group. Similarly an *existential sentence* is one of the form $\exists\overline{x}\{\phi(\overline{x})\}$ where $\overline{x}$ and $\phi(\overline{x})$ are as above. For example

$$\exists(x, y)\{xy \neq yx\}$$

is an existential sentence describing a non-Abelian group. A *universal-existential sentence* is one of the form $\forall\overline{x}\exists\overline{y}\{\phi(\overline{x}, \overline{y})\}$. Similarly defined is an *existential-universal sentence*. It is known that every sentence of L_0 is logically equivalent to one of the form $Q_1x_1 \ldots Q_nx_n\phi(\overline{x})$ where $\overline{x} = (x_1, \ldots, x_n)$ is a tuple of distinct variables, each Q_i for $i = 1, \ldots, n$ is a quantifier, either $\forall$ or $\exists$, and $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing freely at most the variables $x_1, \ldots, x_n$. Further vacuous quantifications are permitted. Finally a *positive sentence* is one logically equivalent to a sentence constructed using (at most) the connectives $\vee, \wedge, \forall, \exists$.

If G is a group then the *universal theory* of G consists of the set of all universal sentences of L_0 true in G. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same *existential theory*. The set of all sentences of L_0 true in G is called the *first-order theory* or the *elementary theory* of G. We denote this by $Th(G)$. We note that being *first-order* or *elementary* means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values only individual group elements – never, for example, subsets of, nor functions on, the group in which they are interpreted.

We say that two groups G and H are *elementarily equivalent* (symbolically $G \equiv H$) if they have the same first-order theory, that is $Th(G) = Th(H)$.

Group monomorphisms which preserve first-order formulas are called elementary embeddings. Specifically, if H and G are groups and

$$f : H \to G$$

is a monomorphism then f is an *elementary embedding* provided whenever $\phi(x_0, \ldots, x_n)$ is a formula of L_0 containing free at most the distinct variables $x_0, \ldots, x_n$ and $(h_0, \ldots, h_n) \in H^{n+1}$ then $\phi(h_0, \ldots, h_n)$ is true in H if and only if $\phi(f(h_0), \ldots, f(h_n))$ is true in G. If H is a subgroup of G and the inclusion map $i: H \to G$ is an elementary embedding then we say that G is an *elementary extension* of H.

The genesis of the Tarski problems is the observation that most theorems concerning free groups are independent of the rank of the free group. As an example we note the Nielsen–Schreier Theorem (see Chapter 2) which says that any subgroup of a free group is itself a free group (independent of the rank of the overgroup). Another example is the result that an Abelian subgroup of a free group, again of any rank, must be cyclic. Proceeding further, suppose that $n < m$ are positive integers. From the Nielsen–Schreier Theorem it is clear that a free group of rank n can be embedded isomorphically into a free group of rank m. Hence F_n can be embedded into F_m. Further it can be shown that a free group of any countable rank can be embedded isomorphically into a free group of rank 2. It follows that F_m can be embedded into F_n. Therefore F_n and F_m must be *similar*.

There was some initial early success on the Tarski conjectures. Vaught showed that the Tarski conjectures 1,2 are true if G and H are both free groups of infinite rank. Combining his result with the elementary chain theorem (see Chapter 4) reduced the conjectures to free groups of finite rank. He also provided a criterion, now called the Tarski–Vaught criterion, to determine if an embedding of one group into another is an elementary embedding.

The next significant progress was due to Merzljakov. The *positive theory* of a group G consists of all the positive sentences true in G. Merzljakov [185] showed that the non-Abelian free groups have the same positive theory. A proof of Merzljakov's [185] result can be given involving generalized equations and a quantifier elimination process. This was a precursor to the methods used in the eventual solution of the overall Tarski problems.

Work following Merzljakov centered on restricted theories of free groups. It is fairly straightforward to show that any two non-Abelian free groups satisfy the same universal theory. Sacerdote [226] proved that this could be extended to *universal-existential sentences* or π_2-sentences (see Chapter 4).

Despite this early sucessful work, the conjectures remained open for over fifty years after Tarski initially proposed them. In a 1988 paper surveying combinatorial group theory [175] Roger Lyndon called the Tarski problems, which he described as folklore, among the outstanding open problems (at that time) in the field. After this point the pieces in the big puzzle began to placed together. First a result of Gaglione and Spellman and independently V. Remeslennikov and building on results of Gilbert Baumslag and Benjamin Baumslag showed that finitely generated groups that share the same universal theory as the class of non-Abelian free groups are precisely the class of finitely generated fully residually free groups. This shifted the focus to the class of finitely generated fully residually free groups in the search for the class of groups that share the same complete first-order theory as the non-Abelian free groups. Further from the Tarski–Vaught criterion the concentration was on the solution of equations in groups.

Dealing with systems of equation over free groups, it was clear from the beginning that to deal with the Tarski's conjectures a precise description of solution sets of equations (and inequations) over free groups was needed. Therefore in analogy with the classical solutions of polynomial equations over fields what was needed was a translation of classicial algebraic geometry to an algebraic geometry over groups. In the late 1990s Gilbert Baumslag, Olga Kharlampovich, Alexei Myasnikov, and Vladimir Remeslennikov developed the basics of algebraic geometry over groups introducing analogs of the standard algebraic geometric notions such as algebraic sets, the Zariski topology, Noetherian domains, irreducible varieties, radicals and coordinate groups.

The first general results on equations in groups appeared in the 1960s. Roger Lyndon developed several extremely important ideas. He considered completions of a given group G by allowing exponents in various rings and then used these completions to parameterize solutions of equations in G. Along these lines he introduced the free exponential group $F^{\mathbb{Z}[t]}$ with exponents in the integral polynomial ring $\mathbb{Z}[t]$. Subsequently it was shown that the finitely generated subgroups of this free exponential group coincides with the class of finitley generated fully residually free groups and hence with the class of universally free groups.

Another idea originating with Lyndon, in addition to generalizing the ring of exponents to $\mathbb{Z}[t]$, is to consider groups with free length functions with values in some ordered Abelian group. This allows one to axiomatize the classical Nielsen technique based on the standard length function in free groups and apply it to "non-standard" extensions of free groups, for instance, to ultrapowers of free groups. A link with the Tarski problems comes here by the Keisler–Shelah theorem, that states that two groups are elementarily equivalent if and only if their ultrapowers (with respect to a non-principal ultrafilter) are isomorphic.

In the eighties new crucial concepts were introduced. Makanin proved the algorithmic decidability of the Diophantine problem over free groups, and showed that both, the universal theory and the positive theory of a free group are algorithmically decidable. He created an extremely powerful technique (the Makanin elimination process) to deal with equations over free groups. Shortly afterwards, Razborov then described the solution set of an arbitrary system of equations over a free group in terms of what is known now as Makanin–Razborov diagrams.

A few years later Edmunds and Commerford and Grigorchuck and Kurchanov described solution sets of arbitrary quadratic equations over free groups. These equations came to group theory from topology and their role in group theory was not initially clear. However it was subsequently proved, and it became fundamental to the proof of the Tarski conjectures, that an arbitrary system of equations is equivalent to a collection of a special type of quadratic systems.

This book is laid out in the following manner. In Chapter 2 we present the necessary material from Combinatorial Group Theory. This will include the material on free groups and group amalgams.

Over the past twenty years, building on work of Gromov, Rips, Bass and Serre and others, geometric ideas have gained prominence. This has been given the overall name Geometric Group Theory and includes hyperbolic group theory and the theory of groups acting on various types of trees. In Chapter 3 we describe the essential ideas in Geometric Group Theory.

In Chapter 4 we will formally introduce the ideas from first-order languages and model theory, most of which are not as well-known to group theorists as they should be. We will also review the concepts of filters, ultra-filters and ultra-products which are essential tools in the study of elementary properties.

In Chapter 5 we will give a more formal description of the Tarski problems as well as a survey of Tarski-like results for other classes of groups.

In Chapters 6 and 7 we describe the vast body of results on fully residually free groups. In Chapter 6 we introduce fully residually free groups, and related concepts, and present the basic properties of such groups including the equivalence with universally free groups. We also describe the general structure theory of these groups.

In Sela's approach the class of fully residually free groups arises as the class of limiting groups from homomorphisms into free groups. Sela terms these limit groups. In Chapter 7 we describe the equivalences for various interpretations of the class of limit groups including some topological interpretations.

In Chapter 8 we present the basic framework of algebraic geometry over groups. This includes algebraic sets, the Zariski topology, Noetherian domains, irreducible varieties, radicals and coordinate groups. We also prove that the coordinate groups of irreducible algebraic varieties over free groups are the limit groups.

In Chapter 9 we outline the Kharlampovich–Myasnikov proof of the Tarski problems. This involves an induction on the number of quantifiers in a logical sentence and a quantifier elimination process that they call the elimination process.

As part of the solution of the Tarski conjectures both Kharlampovich–Myasnikov and Sela provide a complete description of the class of groups that share the elementary theory of the non-Abelian free groups. These extend beyond the class of non-Abelian free groups and are called the elementary free groups. In Chapter 10 we consider the general theory of the elementary free groups and present some properties that go beyond what is true in free groups. It is known that the surface groups of high enough genus are elementary free and this provides a method to prove results in surface groups that are otherwise quite inaccesible and difficult. We discuss this also in Chapter 10.

Finally in Chapter 11 we discuss a large body of results concerning discriminating groups, a class of groups introduced by G.Baumslag, Myasnikov and Remeslennikov as a by-product of the development of algebraic geometry.

We hope that the book will find good use among the group theoretic and logic community.

We would like to thank people who have looked over portions of the book and helped us with the preparation including Gilbert Baumslag and Olga Kharlampovich. Especially we thank Anja Moldenhauer who carefully looked over the chapters and prepared the diagrams and figures.

Ben Fine
Anthony Gaglione
Alexei Myasnikov
Gerhard Rosenberger
Dennis Spellman

Contents

1 Group theory and logic: introduction

1.1 Group theory and logic

The algebraic concept of a group and eventually the discipline of group theory arose in the early nineteenth century initially from the solution by Galois to the problem of solvability by radicals. This early work led primarily to finite groups and specifically to permutation groups, although as Cayley's Theorem points out this is really no restriction. Later, infinite groups became prominent through their use in Geometry and Klein's Erlanger Program (1876). Continuous groups were introduced by Lie and others to extend the methods of Galois for algebraic equations to the solutions of differential equations. Almost concurrently with the introduction of continuous groups, infinite discrete groups arose as tools in the study of combinatorial topology as introduced by Poincare. In addition, infinite discrete groups also became prominent in complex analysis via the work of Fricke and Klein on discrete groups of motions of the hyperbolic plane (Fuchsian groups). As these group objects were introduced through algebra, topology, geometry and analysis it became clear that there were strong interactions with formal logic. Each of the concrete examples of groups, mentioned above, permutation groups, matrix groups, groups of geometric transformations etc. are *models* in the sense of formal logic of abstract logical structures and languages. These ties became even clearer in the twentieth century as the study of mathematical logic became formalized.

This book will concentrate on the interactions between group theory and logic and will focus primarily on infinite discrete groups. We will deal with ideas and extensions of concepts arising around the *Tarski Problems* and their solution. The statements of the Tarski problems will be explained in the next section. First however we introduce some material that is needed to describe these problems.

The study of infinite discrete group theory essentially uses *combinatorial group theory*. This subdiscipline, within group theory, can roughly be described as the study of groups via *group presentations*. A presentation for a group G consists of a set of generators $\{g_\nu\}$ for G from which any element of G can be generated as a word or expression in the $\{g_\nu\}$ together with a set of relations on these generators from which any part of the group table can be constructed. In Chapter 2 we will examine combinatorial group theory in detail. Although a group presentation is a succinct way to express a group, it was clear from the beginnings of the discipline, that working with group presentations required some detailed algorithmic knowledge and certain decision questions.

In 1910 Max Dehn, as part of his work with group presentations for the fundamental groups of orientable surfaces, presented the three most *fundamental decision problems*. The first of these is the *word problem* or *identity problem*. This is given as follows:

(1) Word Problem: Suppose G is a group given by a finite presentation. Does there exist an algorithm to determine if an arbitrary word W in the generators of G defines the identity element of G?

Specifically if

$$G = \langle g_i; i = 1, \ldots, n; R_j = 1, j = 1, \ldots, m\rangle$$

is a finite presentation for G and $W(g_\nu)$ is an arbitrary word in the generators of G, can one decide algorithmically, in a finite number of steps, whether $W(g_\nu)$ represents the identity in G or not. If such an algorithm exists we say that G has a *solvable word problem*. If not, G has an *unsolvable word problem*. Dehn presented a geometric method to show that the fundamental group of an orientable surface of genus $g \geq 2$, which we denote by S_g, has a solvable word problem. In particular he gave an algorithm which systematically reduced the length of any word equal to the identity in $\pi(S_g)$. If a particular word's length is greater than 1 and cannot be reduced then that word does not represent the identity. Such an algorithm is now called a *Dehn algorithm*. Subsequently small cancellation theory, (see Chapter 3), was developed to determine additional groups that have Dehn algorithms. More recently it was shown that finitely presented groups with Dehn algorithms are precisely the word-hyperbolic groups of Gromov (see Chapter 2). In 1955 Novikov [205] and independently Boone [34] proved that, in general. the word problem is unsolvable, that is, there exist finitely presented groups with unsolvable word problems. Hence questions about word problems now focus on which particular classes of groups have solvable word problems. As decribed by Magnus (see [178]), given the Novikov–Boone result, any solution of the word problem is actually a triumph over nature.

The second fundamental decision problem is the *conjugacy problem* given by:

(2) Conjugacy Problem: Suppose G is a group given by a finite presentation. Does there exist an algorithm to determine if an arbitary pair of words u, v in the generators of G define conjugate elements of G?

A solution of the conjugacy problem implies a solution of the word problem. Hence it follows from the Novikov–Boone result that the conjugacy problem is unsolvable in general. It has been shown further (see [178] and [186]) that there do exist finitely presented groups with solvable word problems and unsolvable conjugacy problems. As for the word problem, results on the conjugacy problem, concentrate on which groups, or classes of groups, do have solvable conjugacy problems. In particular many small cancellation groups also have solvable conjugacy problems. Geometric techniques were introduced into small cancellation theory to mimic Dehn's original approach and prove these results for small cancellation groups.

The final fundamental decision problem is the most difficult. It is the *isomorphism problem*.

(3) Isomorphism Problem: Does there exist an algorithm to determine given two arbitary finite presentations whether the groups they present are isomorphic or not?