José G. Vargas

DIFFERENTIAL GEOMETRY FOR PHYSICISTS AND MATHEMATICIANS

Moving Frames and Differential Forms: From Euclid Past Riemann

World Scientific

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PST Associates, LLC, USA

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Preface

The principle that informs this book. This is a book on differential geometry that uses the method of moving frames and the exterior calculus throughout. That may be common to a few works. What is special about this one is the following. After introducing the basic theory of differential forms and pertinent algebra, we study the "flat cases" known as affine and Euclidean spaces, and simple examples of their generalizations. In so doing, we seek understanding of advanced concepts by first dealing with them in simple structures. Differential geometry books often resort to formal definitions of bundles, Lie algebras, etc. that are best understood by discovering them in a natural way in cases of interest. Those books then provide very recondite examples for the illustration of advanced concepts, say torsion, even though very simple examples exist. Misunderstandings ensue.

In 1492 Christopher Columbus crossed the Atlantic using an affine connection in a simplified form (a connection is nothing but a rule to navigate a manifold). He asked the captains of the other two ships in his small flotilla to always maintain what he considered to be the same direction: West. That connection has torsion. Élie Cartan introduced it in the mathematical literature centuries later [13]. We can learn connections from a practical point of view, the practical one of Columbus. That will help us to easily understand concepts like frame bundle, connection, valuedness, Lie algebra, etc., which might otherwise look intimidating. Thus, for example, we shall slowly acquire a good understanding of affine connections as differential 1−forms in the affine frame bundle of a differentiable manifold taking values in the Lie algebra of the affine group and having such and such properties. Replace the term affine with the terms Euclidean, conformal, projective, etc. and you have entered the theories of Euclidean, projective, conformal ... connections.

Cartan's versus the modern approach to geometry. It is sometimes stated that É. Cartan's work was not rigorous, and that it is not possible to make it so. This statement has led to the development of other methods to do differential geometry, full of definitions and distracting concepts; not the style that physicists like.

Yeaton H. Clifton was a great differential topologist, an opinion of this author which was also shared by the well known late mathematician S.-S. Chern in private conversation with this author. Clifton had once told me that the only thing that was needed to make rigorous Cartan's theory of connections was to add a couple of definitions. A few years later, upon the present author's prodding, Clifton delivered on his claim. To be precise, he showed that just a major definition and a couple of theorems were needed. The proof is in the pudding. It is served in the last section of chapter 8 and in the second section of chapter 9.

Unfortunately, Cartan's approach has virtually vanished from the modern literature. Almost a century after his formulation of the theory of affine and Euclidean connections as a generalization of the geometry of affine and Euclidean spaces [11], [12], [14], an update is due on his strategy for the study of generalized spaces with the method of the moving frame [20]. We shall first study from the perspective of bundles and integrability of equations two flat geometries (their technical name is Klein geometries) and then proceed with their Cartan generalization. In those Klein geometries, affine and Euclidean, concepts like equations of structure already exist, and the mathematical expression of concepts like curvature and torsion already arise in full-fledged form. It simply happens that they take null values.

Mathematical substance underlying the notation. There is a profound difference between most modern presentations and ours. Most authors try to fit everything that transforms tensorially into the mold of $(p, q)-$tensors (p times contravariant and q times covariant). Following Kähler in his generalization of Cartan's calculus, [46], [47], [48], we do not find that to be the right course of action. Here is why.

Faced with covariant tensor fields that are totally skew-symmetric, the modern approach that we criticize ignores that the natural derivative of a tensor field, whether skew-symmetric or not, is the covariant derivative. They resort to exterior derivatives, which belong to exterior algebra. That is unnatural and only creates confusion. Exterior differentiation should be applied only to exterior differential forms, and these are not skew-symmetric tensors. They only look that way.

Covariant tensor fields have subscripts, but so do exterior differential forms. For most of the authors that we criticize, the components of those two types of mathematical objects have subscripts, which they call q indices. But not all the q indices are born equal. There will be skew-symmetry and exterior differentiation in connection with some of them —"differential form" subscripts— but not in connection with the remaining ones, whether they are skew-symmetric with respect to those indices or not. They are tensor subscripts. Like superscripts, they are associated with covariant differentiation.

Correspondingly, the components of quantities in the Cartan and Kähler calculus have —in addition to a series of superscripts— two series of subscripts, one for integrands and another one for multilinear functions of vectors. This is explicitly exhibited in Kähler [46], [47], [48].

The paragon of quantities with three types of indices. Affine curvature is a $(1, 1)-$tensor-valued differential $2-$form. The first "1" in the pair is for a superscript, and the other one is for a subscript. Torsions are $(1, 0)-$valued differential $2-$forms and contorsions are $(1, 1)-$valued differential $1-$forms.

Let $\mathbf{v}$ represent vector fields and let d be the operator that Cartan calls exterior differentiation. $d\mathbf{v}$ is a vector-valued differential 1−form, and $dd\mathbf{v}$ is a vector-valued differential 2−form. Experts not used to Cartan's notation need be informed that $dd\mathbf{v}$ is $(v^{\mu}R^{\nu}_{\mu\lambda_1\lambda_2})\omega^{\lambda_1} \wedge \omega^{\lambda_2}\mathbf{e}_{\nu}$. Relative to bases of $(p = 1, q = 0)$−valued differential 2−forms, the components of $dd\mathbf{v}$ are $(v^{\mu}R^{\nu}_{\mu\lambda_1\lambda_2})$. One can then define a $(1, 1)$−valued differential 2−form whose components are the $R^{\nu}_{\mu\lambda_1\lambda_2}$'s, and whose evaluation on $\mathbf{v}$ (responding to the $q = 1$ part of the valuedness) yields $dd\mathbf{v}$. Hence, the traditional (p, q)−characterization falls short of the need for a good understanding of issues concerned with the curvature differential form.

Bundles are of the essence. The perspective of valuedness that we have just mentioned is one which best fits sections of frame bundles, and transformations relating those sections. Lest be forgotten, the set of all inertial frames (they do not need to be inertial, but that is the way in which they appear in the physics literature) constitutes a frame bundle. Grossly speaking, a bundle is a set whose elements are organized like those inertial frames are. The ones at any given point constitute the fiber at that point. We have identical fibers at different points. There must be a group acting in the bundle (like Poincaré's is in our example), and a subgroup acting in the fibers (the homogenous Lorentz group in our example).

An interesting example of section of a bundle is found in cosmology. One is computing in a particular section when one refers quantities to the frame of reference of matter at rest in the large.

A section is built with one and only one frame from each fiber, the choice taking place in a continuous way. But, for foundational purposes, it is better to think in terms of the bundle than of the sections. At an advanced level, one speaks of Lie algebra valuedness of connections, the Lie algebra being a vector space of the same dimension as the bundle. All this is much simpler than it sounds when one really understands Euclidean space. We will.

It is unfortunate that books on the geometry of physics deal with connections valued in Lie algebras pertaining to auxiliary bundles (i.e. not directly related to the tangent vectors) and do not even bother with the Lie algebras of bundles of frames of tangent vectors. Which physicist ever mentions what is the Lie algebra where the Levi-Civita connection takes its values? Incidentally, the tangent vectors themselves constitute a so called fiber bundle, each fiber being constituted by all tangent vectors at any given point. It is the tangent bundle.

This author claims that the geometry of groups such as $SU(3)$ and $U(1) \times SU(2)$ fits in appropriately extended tangent bundle geometry, if one just knows where to look. One does not need auxiliary bundles. That will not be dealt with in this book, but in coming papers. This book will tell you whether I deserve your trust and should keep following me where I think that the ideas of Einstein, Cartan and Kähler take us.

Assume there were a viable option of relating $U(1) \times SU(2) \times SU(3)$ to bundles of tangent vectors, their frames, etc. It would be unreasonable to remain satisfied with auxiliary bundles (Yang-Mills theory). In any case, one should understand "main bundles geometry" (i.e. directly related to the tangent bundle) before studying and passing judgement on the merits and dangers of Yang-Mills theory.

Specific features distinguishing this book are as follows:

1. Differential geometry is presented from the perspective of integrability, using so called moving frames in frame bundles. The systems of differential equations in question emerge in the study of affine and Euclidean Klein geometries, those specific systems being integrable.

2. In this book, it does not suffice whether the equations of the general case (curved) have the appropriate flat limit. It is a matter of whether we use in the general case concepts which are the same or as close as possible to the intuitive concepts used in flat geometry. Thus, the all-pervasive definition of tangent vectors as differential operators in the modern literature is inimical to our treatment.

3. In the same spirit of facilitating understanding by non-mathematicians, differential forms are viewed as functions of curves, surfaces and hypersurfaces [65] (We shall use the term hypersurface to refer to manifolds of arbitrary dimension that are not Klein spaces). In other words, they are not skew-symmetric multilinear functions of vectors but cochains.

This book covers almost the same material as a previous book by this author [85] *except for the following*:

1. The contents of chapters 1, 3 and 12 has been changed or extended very significantly.

2. We have added the appendices. Appendix A presents the classical theory of curves and surfaces, but treated in a totally novel way through the introduction of the concept of canonical frame field of a surface (embedded in 3-D Euclidean space). We could have made it into one more chapter, but we have not since connections connect tangent vectors in the book except in that appendix; vectors in 3-D Euclidean space that are not tangent vectors to the specific curves and surfaces being considered are nevertheless part of the subject matter.

Appendix B speaks of the work of the mathematical geniuses Élie Cartan and Hermann Grassmann, in order to honor the enormous presence of their ideas in this book. Appendix C is the list of publications of this author for those who want to deal further into topics not fully addressed in this book but directly related to it. You can find there papers on Finsler geometry, unification with teleparallelism, the Kähler calculus, alternatives to the bundle of orthonormal frames, etc.

3. Several sections have been added at the end of several chapters, touching subjects such as diagonalization of metrics and orthonormalization of frames, Clifford and Lie algebras, etc.

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