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Differential Analysis on Complex Manifolds

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PREFACE TO THE FIRST EDITION

This book is an outgrowth and a considerable expansion of lectures given at Brandeis University in 1967–1968 and at Rice University in 1968–1969. The first four chapters are an attempt to survey in detail some recent developments in four somewhat different areas of mathematics: geometry (manifolds and vector bundles), algebraic topology, differential geometry, and partial differential equations. In these chapters, I have developed various tools that are useful in the study of compact complex manifolds. My motivation for the choice of topics developed was governed mainly by the applications anticipated in the last two chapters. Two principal topics developed include Hodge's theory of harmonic integrals and Kodaira's characterization of projective algebraic manifolds.

This book should be suitable for a graduate level course on the general topic of complex manifolds. I have avoided developing any of the theory of several complex variables relating to recent developments in Stein manifold theory because there are several recent texts on the subject (*Gunning and Rossi, Hörmander*). The text is relatively self-contained and assumes familiarity with the usual first year graduate courses (including some functional analysis), but since geometry is one of the major themes of the book, it is developed from first principles.

Each chapter is prefaced by a general survey of its content. Needless to say, there are numerous topics whose inclusion in this book would have been appropriate and useful. However, this book is not a treatise, but an attempt to follow certain threads that interconnect various fields and to culminate with certain key results in the theory of compact complex manifolds. In almost every chapter I give formal statements of theorems which are understandable in context, but whose proof oftentimes involves additional machinery not developed here (e.g., the Hirzebruch Riemann-Roch Theorem); hopefully, the interested reader will be sufficiently prepared (and perhaps motivated) to do further reading in the directions indicated.

Text references of the type (4.6) refer to the 6th equation (or theorem, lemma, etc.) in Sec. 4 of the chapter in which the reference appears. If the reference occurs in a different chapter, then it will be prefixed by the Roman numeral of that chapter, e.g., (II.4.6.).

I would like to express appreciation and gratitude to many of my colleagues and friends with whom I have discussed various aspects of the book during its development. In particular I would like to mention M. F. Atiyah, R. Bott, S. S. Chern, P. A. Griffiths, R. Harvey, L. Hörmander, R. Palais, J. Polking, O. Riemenschneider, H. Rossi, and W. Schmid whose comments were all very useful. The help and enthusiasm of my students at Brandeis and Rice during the course of my first lectures, had a lot to do with my continuing the project. M. Cowen and A. Dubson were very helpful with their careful reading of the first draft. In addition, I would like to thank two of my students for their considerable help. M. Windham wrote the first three chapters from my lectures in 1968–69 and read the first draft. Without his notes, the book almost surely would not have been started. J. Drouilhet read the final manuscript and galley proofs with great care and helped eliminate numerous errors from the text.

I would like to thank the Institute for Advanced Study for the opportunity to spend the year 1970–71 at Princeton, during which time I worked on the book and where a good deal of the typing was done by the excellent Institute staff. Finally, the staff of the Mathematics Department at Rice University was extremely helpful during the preparation and editing of the manuscript for publication.

R. O. Wells, Jr.

Houston
December 1972

PREFACE TO THE SECOND EDITION

In this second edition I have added a new section on the classical finite-dimensional representation theory for $\mathfrak{sl}(2, \mathbb{C})$. This is then used to give a natural proof of the Lefschetz decomposition theorem, an observation first made by S. S. Chern. H. Hecht observed that the Hodge $*$-operator is essentially a representation of the Weyl reflection operator acting on $\mathfrak{sl}(2, \mathbb{C})$ and this fact leads to new proofs (due to Hecht) of some of the basic Kähler identities which we incorporate into a completely revised Chapter V. The remainder of the book is generally the same as the first edition, except that numerous errors in the first edition have been corrected, and various examples have been added throughout.

I would like to thank my many colleagues who have commented on the first edition, which helped a great deal in getting rid of errors. Also, I would like to thank the graduate students at Rice who went carefully through the book with me in a seminar. Finally, I am very grateful to David Yingst and David Johnson who both collated errors, made many suggestions, and helped greatly with the editing of this second edition.

Houston
July 1979

R. O. Wells, Jr.

CONTENTS

CHAPTER I

MANIFOLDS AND VECTOR BUNDLES

There are many classes of manifolds which are under rather intense investigation in various fields of mathematics and from various points of view. In this book we are primarily interested in *differentiable manifolds* and *complex manifolds*. We want to study (a) the "geometry" of manifolds, (b) the analysis of functions (or more general objects) which are defined on manifolds, and (c) the interaction of (a) and (b). Our basic interest will be the application of techniques of real analysis (such as differential geometry and differential equations) to problems arising in the study of complex manifolds. In this chapter we shall summarize some of the basic definitions and results (including various examples) of the elementary theory of manifolds and vector bundles. We shall mention some nontrivial embedding theorems for differentiable and real-analytic manifolds as motivation for Kodaira's characterization of projective algebraic manifolds, one of the principal results which will be proved in this book (see Chap. VI). The "geometry" of a manifold is, from our point of view, represented by the behavior of the tangent bundle of a given manifold. In Sec. 2 we shall develop the concept of the tangent bundle (and derived bundles) from, more or less, first principles. We shall also discuss the continuous and C^∞ classification of vector bundles, which we shall not use in any real sense but which we shall meet a version of in Chap. III, when we study Chern classes. In Sec. 3 we shall introduce almost complex structures and the calculus of differential forms of type (p, q), including a discussion of integrability and the Newlander-Nirenberg theorem.

General background references for the material in this chapter are Bishop and Crittenden [1], Lang [1], Narasimhan [1], and Spivak [1], to name a few relatively recent texts. More specific references are given in the individual sections. The classical reference for calculus on manifolds is de Rham [1]. Such concepts as differential forms on differentiable manifolds, integration on chains, orientation, Stokes' theorem, and partition of unity are all covered adequately in the above references, as well as elsewhere, and in this book we shall assume familiarity with these concepts, although we may review some specific concept in a given context.

1. Manifolds

We shall begin this section with some basic definitions in which we shall use the following standard notations. Let **R** and **C** denote the fields of real and complex numbers, respectively, with their usual topologies, and let K denote either of these fields. If D is an open subset of K^n, we shall be concerned with the following function spaces on D:

(a) $K = \mathbf{R}$:

(1) $\mathcal{E}(D)$ will denote the real-valued *indefinitely differentiable* functions on D, which we shall simply call C^∞ functions on D; i.e., $f \in \mathcal{E}(D)$ if and only if f is a real-valued function such that partial derivatives of all orders exist and are continuous at all points of D [$\mathcal{E}(D)$ is often denoted by $C^\infty(D)$].

(2) $\mathcal{A}(D)$ will denote the real-valued *real-analytic functions* on D; i.e., $\mathcal{A}(D) \subset \mathcal{E}(D)$, and $f \in \mathcal{A}(D)$ if and only if the Taylor expansion of f converges to f in a neighborhood of any point of D.

(b) $K = \mathbf{C}$:

(1) $\mathcal{O}(D)$ will denote the complex-valued *holomorphic functions* on D, i.e., if $(z_1, \ldots, z_n)$ are coordinates in $\mathbf{C}^n$, then $f \in \mathcal{O}(D)$ if and only if near each point $z^0 \in D$, f can be represented by a convergent power series of the form

$$f(z) = f(z_1, \ldots, z_n) = \sum_{\alpha_1, \ldots, \alpha_n = 0}^{\infty} a_{\alpha_1, \ldots, \alpha_n}(z_1 - z_1^0)^{\alpha_1} \cdots (z_n - z_n^0)^{\alpha_n}.$$

(See, e.g., Gunning and Rossi [1], Chap. I, or Hörmander [2], Chap. II, for the elementary properties of holomorphic functions on an open set in $\mathbf{C}^n$). These particular classes of functions will be used to define the particular classes of manifolds that we shall be interested in.

A *topological n-manifold* is a Hausdorff topological space with a countable basis† which is locally homeomorphic to an open subset of $\mathbf{R}^n$. The integer n is called the *topological dimension* of the manifold. Suppose that $\mathcal{S}$ is one of the three K-valued families of functions defined on the open subsets of K^n described above, where we let $\mathcal{S}(D)$ denote the functions of $\mathcal{S}$ defined on D, an open set in K^n. [That is, $\mathcal{S}(D)$ is either $\mathcal{E}(D)$, $\mathcal{A}(D)$, or $\mathcal{O}(D)$. We shall only consider these three examples in this chapter. The concept of a family of functions is formalized by the notion of a *presheaf* in Chap. II.]

Definition 1.1: An $\mathcal{S}$-*structure*, $\mathcal{S}_M$, on a k-manifold M is a family of K-valued continuous functions defined on the open sets of M such that

†The additional assumption of a countable basis ("countable at infinity") is important for doing analysis on manifolds, and we incorporate it into the definition, as we are less interested in this book in the larger class of manifolds.

(a) For every $p \in M$, there exists an open neighborhood U of p and a homeomorphism $h : U \longrightarrow U'$, where U' is open in K^n, such that for any open set $V \subset U$

$$f : V \longrightarrow K \in \mathcal{S}_M \text{ if and only if } f \circ h^{-1} \in \mathcal{S}(h(V)).$$

(b) If $f : U \rightarrow K$, where $U = \cup_i U_i$ and U_i is open in M, then $f \in \mathcal{S}_M$ if and only if $f|_{U_i} \in \mathcal{S}_M$ for each i.

It follows clearly from (a) that if $K = \mathbf{R}$, the dimension, k, of the topological manifold is equal to n, and if $K = \mathbf{C}$, then $k = 2n$. In either case n will be called the *K-dimension* of M, denoted by $\dim_K M = n$ (which we shall call *real-dimension* and *complex-dimension*, respectively). A manifold with an $\mathcal{S}$-structure is called an $\mathcal{S}$*-manifold*, denoted by $(M, \mathcal{S}_M)$, and the elements of $\mathcal{S}_M$ are called $\mathcal{S}$*-functions* on M. An open subset $U \subset M$ and a homeomorphism $h : U \longrightarrow U' \subset K^n$ as in (a) above is called an $\mathcal{S}$*-coordinate system*.

For our three classes of functions we have defined

(a) $\mathcal{S} = \mathcal{E}$: *differentiable* (or C^∞) *manifold*, and the functions in $\mathcal{E}_M$ are called C^∞ *functions* on open subsets of M.

(b) $\mathcal{S} = \mathcal{A}$: *real-analytic manifold*, and the functions in $\mathcal{A}_M$ are called *real-analytic functions* on open subsets of M.

(c) $\mathcal{S} = \mathcal{O}$: *complex-analytic* (or simply *complex*) *manifold*, and the functions in $\mathcal{O}_M$ are called *holomorphic* (or *complex-analytic functions*) on M.

We shall refer to $\mathcal{E}_M$, $\mathcal{A}_M$, and $\mathcal{O}_M$ as *differentiable*, *real-analytic*, and *complex structures* respectively.

Definition 1.2:

(a) An $\mathcal{S}$*-morphism* $F : (M, \mathcal{S}_M) \longrightarrow (N, \mathcal{S}_N)$ is a continuous map, $F : M \longrightarrow N$, such that

$$f \in \mathcal{S}_N \text{ implies } f \circ F \in \mathcal{S}_M.$$

(b) An $\mathcal{S}$*-isomorphism* is an $\mathcal{S}$-morphism $F: (M, \mathcal{S}_M) \rightarrow (N, \mathcal{S}_N)$ such that $F : M \longrightarrow N$ is a homeomorphism, and

$$F^{-1} : (N, \mathcal{S}_N) \longrightarrow (M, \mathcal{S}_M) \text{ is an } \mathcal{S}\text{-morphism.}$$

It follows from the above definitions that if on an $\mathcal{S}$-manifold $(M, \mathcal{S}_M)$ we have two coordinate systems $h_1 : U_1 \longrightarrow K^n$ and $h_2 : U_2 \longrightarrow K^n$ such that $U_1 \cap U_2 \neq \varnothing$, then

(1.1) $h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \longrightarrow h_2(U_1 \cap U_2)$ is an $\mathcal{S}$-isomorphism on open subsets of $(K^n, \mathcal{S}_{K^n})$.

Conversely, if we have an open covering $\{U_\alpha\}_{\alpha \in A}$ of M, a topological manifold, and a family of homeomorphisms $\{h_\alpha: U_\alpha \longrightarrow U'_\alpha \subset K^n\}_{\alpha \in A}$ satisfying (1.1), then this defines an $\mathcal{S}$-structure on M by setting $\mathcal{S}_M = \{f: U \longrightarrow K\}$ such that U is open in M and $f \circ h_\alpha^{-1} \in \mathcal{S}(h_\alpha(U \cap U_\alpha))$ for all $\alpha \in A$; i.e., the functions in $\mathcal{S}_M$ are pullbacks of functions in $\mathcal{S}$ by the homeomorphisms $\{h_\alpha\}_{\alpha \in A}$. The collection $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ is called an *atlas* for $(M, \mathcal{S}_M)$.

In our three classes of functions, the concept of an $\mathcal{S}$-morphism and $\mathcal{S}$-isomorphism have special names:

(a) $\mathcal{S} = \mathcal{E}$: *differentiable mapping* and *diffeomorphism* of M to N.

(b) $\mathcal{S} = \mathcal{A}$: *real-analytic mapping* and *real-analytic isomorphism* (or *bianalytic* mapping) of M to N.

(c) $\mathcal{S} = \mathcal{O}$: *holomorphic mapping* and *biholomorphism* (*biholomorphic mapping*) of M to N.

It follows immediately from the definition above that a differentiable mapping

$$f: M \longrightarrow N,$$

where M and N are differentiable manifolds, is a continuous mapping of the underlying topological space which has the property that in local coordinate systems on M and N, f can be represented as a matrix of C^∞ functions. This could also be taken as the definition of a differentiable mapping. A similar remark holds for the other two categories.

Let N be an arbitrary subset of an $\mathcal{S}$-manifold M; then an *$\mathcal{S}$-function on N* is defined to be the restriction to N of an $\mathcal{S}$-function defined in some open set containing N, and $\mathcal{S}_M|_N$ consists of all the functions defined on relatively open subsets of N which are restrictions of $\mathcal{S}$-functions on the open subsets of M.

Definition 1.3: Let N be a closed subset of an $\mathcal{S}$-manifold M; then N is called an *$\mathcal{S}$-submanifold* of M if for each point $x_0 \in N$, there is a coordinate system $h: U \longrightarrow U' \subset K^n$, where $x_0 \in U$, with the property that $h|_{U \cap N}$ is mapped onto $U' \cap K^k$, where $0 \leq k \leq n$. Here $K^k \subset K^n$ is the standard embedding of the linear subspace K^k into K^n, and k is called the *K-dimension* of N, and $n - k$ is called the *K-codimension* of N.

It is easy to see that an $\mathcal{S}$-submanifold of an $\mathcal{S}$-manifold M is itself an $\mathcal{S}$-manifold with the $\mathcal{S}$-structure given by $\mathcal{S}_M|_N$. Since the implicit function theorem is valid in each of our three categories, it is easy to verify that the above definition of submanifold coincides with the more common one that an $\mathcal{S}$-submanifold (of k dimensions) is a closed subset of an $\mathcal{S}$-manifold M which is locally the common set of zeros of $n - k$ $\mathcal{S}$-functions whose Jacobian matrix has maximal rank.

It is clear that an n-dimensional complex structure on a manifold induces a $2n$-dimensional real-analytic structure, which, likewise, induces a $2n$-dimensional differentiable structure on the manifold. One of the questions we shall be concerned with is how many different (i.e., nonisomorphic) complex-analytic structures induce the same differentiable structure on a given manifold? The analogous question of how many different differentiable structures exist on a given topological manifold is an important problem in differential topology.

What we have actually defined is a category wherein the objects are $\mathcal{S}$-manifolds and the morphisms are $\mathcal{S}$-morphisms. We leave to the reader the proof that this actually is a category, since it follows directly from the definitions. In the course of what follows, then, we shall use three categories—the differentiable ($\mathcal{S} = \mathcal{E}$), the real-analytic ($\mathcal{S} = \mathcal{A}$), and the holomorphic ($\mathcal{S} = \mathcal{O}$) categories—and the above remark states that each is a subcategory of the former.

We now want to give some examples of various types of manifolds.

Example 1.4 (Euclidean space): K^n, $(\mathbf{R}^n, \mathbf{C}^n)$. For every $p \in K^n$, $U = K^n$ and $h =$ identity. Then $\mathbf{R}^n$ becomes a real-analytic (hence differentiable) manifold and $\mathbf{C}^n$ is a complex-analytic manifold.

Example 1.5: If $(M, \mathcal{S}_M)$ is an $\mathcal{S}$-manifold, then any open subset U of M has an $\mathcal{S}$-structure, $\mathcal{S}_U = \{f|_U : f \in \mathcal{S}_M\}$.

Example 1.6 (Projective space): If V is a finite dimensional vector space over K, then† $\mathbf{P}(V) :=$ {the set of one-dimensional subspaces of V} is called the *projective space* of V. We shall study certain special projective spaces, namely

$$\mathbf{P}_n(\mathbf{R}) := \mathbf{P}(\mathbf{R}^{n+1})$$

$$\mathbf{P}_n(\mathbf{C}) := \mathbf{P}(\mathbf{C}^{n+1}).$$

We shall show how $\mathbf{P}_n(\mathbf{R})$ can be made into a differentiable manifold.

There is a natural map $\pi: \mathbf{R}^{n+1} - \{0\} \longrightarrow \mathbf{P}_n(\mathbf{R})$ given by

$$\pi(x) = \pi(x_0, \ldots, x_n) := \{\text{subspace spanned by } x = (x_0, \ldots, x_n) \in \mathbf{R}^{n+1}\}.$$

The mapping π is onto; in fact, $\pi|_{S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}}$ is onto. Let $\mathbf{P}_n(\mathbf{R})$ have the quotient topology induced by the map π; i.e., $U \subset \mathbf{P}_n(\mathbf{R})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbf{R}^{n+1} - \{0\}$. Hence π is continuous and $\mathbf{P}_n(\mathbf{R})$ is a Hausdorff space with a countable basis. Also, since

$$\pi|_{S^n}: S^n \longrightarrow \mathbf{P}_n(\mathbf{R})$$

is continuous and surjective, $\mathbf{P}_n(\mathbf{R})$ is compact.

If $x = (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} - \{0\}$, then set

$$\pi(x) = [x_0, \ldots, x_n].$$

We say that $(x_0, \ldots, x_n)$ are *homogeneous coordinates* of $[x_0, \ldots, x_n]$. If $(x'_0, \ldots, x'_n)$ is another set of homogeneous coordinates of $[x_0, \ldots, x_n]$,

† $:=$ means that the object on the left is defined to be equal to the object on the right.

then $x_i = tx'_i$ for some $t \in \mathbf{R} - \{0\}$, since $[x_0, \ldots, x_n]$ is the one-dimensional subspace spanned by $(x_0, \ldots, x_n)$ or $(x'_0, \ldots, x'_n)$. Hence also $\pi(x) = \pi(tx)$ for $t \in \mathbf{R} - \{0\}$. Using homogeneous coordinates, we can define a differentiable structure (in fact, real-analytic) on $\mathbf{P}_n(\mathbf{R})$ as follows. Let

$$U_\alpha = \{S \in \mathbf{P}_n(\mathbf{R}) \colon S = [x_0, \ldots, x_n] \text{ and } x_\alpha \neq 0\}, \qquad \text{for } \alpha = 0, \ldots, n.$$

Each U_α is open and $\mathbf{P}_n(\mathbf{R}) = \bigcup_{\alpha=0}^{n} U_\alpha$ since $(x_0, \ldots, x_n) \in \mathbf{R}^{n+1} - \{0\}$. Also, define the map $h_\alpha \colon U_\alpha \longrightarrow \mathbf{R}^n$ by setting

$$h_\alpha([x_0, \ldots, x_n]) = \left(\frac{x_0}{x_\alpha}, \ldots, \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, \ldots, \frac{x_n}{x_\alpha}\right) \in \mathbf{R}^n.$$

Note that both U_α and h_α are well defined by the relation between different choices of homogeneous coordinates. One shows easily that h_α is a homeomorphism and that $h_\alpha \circ h_\beta^{-1}$ is a diffeomorphism; therefore, this defines a differentiable structure on $\mathbf{P}_n(\mathbf{R})$. In exactly this same fashion we can define a differentiable structure on $\mathbf{P}(V)$ for any finite dimensional $\mathbf{R}$-vector space V and a complex-analytic structure on $\mathbf{P}(V)$ for any finite dimensional $\mathbf{C}$-vector space V.

Example 1.7 (Matrices of fixed rank): Let $\mathfrak{M}_{k,n}(\mathbf{R})$ be the $k \times n$ matrices with real coefficients. Let $M_{k,n}(\mathbf{R})$ be the $k \times n$ matrices of rank $k (k \leq n)$. Let $M^m_{k,n}(\mathbf{R})$ be the elements of $\mathfrak{M}_{k,n}(\mathbf{R})$ of rank m $(m \leq k)$. First, $\mathfrak{M}_{k,n}(\mathbf{R})$ can be identified with $\mathbf{R}^{kn}$, and hence it is a differentiable manifold. We know that $M_{k,n}(\mathbf{R})$ consists of those $k \times n$ matrices for which at least one $k \times k$ minor is nonsingular; i.e.,

$$M_{k,n}(\mathbf{R}) = \bigcup_{i=1}^{l} \{A \in \mathfrak{M}_{k,n}(\mathbf{R}) \colon \det A_i \neq 0\},$$

where for each $A \in \mathfrak{M}_{k,n}(\mathbf{R})$ we let $\{A_1, \ldots, A_l\}$ be a fixed ordering of the $k \times k$ minors of A. Since the determinant function is continuous, we see that $M_{k,n}(\mathbf{R})$ is an open subset of $\mathfrak{M}_{k,n}(\mathbf{R})$ and hence has a differentiable structure induced on it by the differentiable structure on $\mathfrak{M}_{k,n}(\mathbf{R})$ (see Example 1.5). We can also define a differentiable structure on $M^m_{k,n}(\mathbf{R})$. For convenience we delete the $\mathbf{R}$ and refer to $M^m_{k,n}$. For $X_0 \in M^m_{k,n}$, we define a coordinate neighborhood at X_0 as follows. Since the rank of X is m, there exist permutation matrices P, Q such that

$$PX_0Q = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix},$$

where A_0 is a nonsingular $m \times m$ matrix. Hence there exists an $\epsilon > 0$ such that $\|A - A_0\| < \epsilon$ implies A is nonsingular, where $\|A\| = \max_{ij} |a_{ij}|$, for $A = [a_{ij}]$. Therefore let

$$W = \{X \in \mathfrak{M}_{k,n} \colon PXQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } \|A - A_0\| < \epsilon\}.$$

Then W is an open subset of $\mathfrak{M}_{k,n}$. Since this is true, $U := W \cap M^m_{k,n}$ is an open neighborhood of X_0 in $M^m_{k,n}$ and will be the necessary coordinate neighborhood of X_0. Note that

$$X \in U \text{ if and only if } D = CA^{-1}B, \qquad \text{where } PXQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

This follows from the fact that

$$\begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_{k-m} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

and

$$\begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_{k-m} \end{bmatrix}$$

is nonsingular (where I_j is the $j \times j$ identity matrix). Therefore

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

have the same rank, but

$$\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

has rank m if and only if $D - CA^{-1}B = 0$.

We see that $M^m_{k,n}$ actually becomes a manifold of dimension $m(n + k - m)$ by defining

$$h\colon U \longrightarrow \mathbf{R}^{m^2 + (n-m)m + (k-m)m} = \mathbf{R}^{m(n+k-m)},$$

where

$$h(X) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathbf{R}^{m(n+k-m)} \qquad \text{for } PXQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

as above. Note that we can define an inverse for h by

$$h^{-1}\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right) = P^{-1}\begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}Q^{-1}.$$

Therefore h is, in fact, bijective and is easily shown to be a homeomorphism. Moreover, if h_1 and h_2 are given as above,

$$h_2 \circ h_1^{-1}\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_2 & B_2 \\ C_2 & 0 \end{bmatrix},$$

where

$$P_2P_1^{-1}\begin{bmatrix} A_1 & B_1 \\ C_1 & C_1A_1^{-1}B_1 \end{bmatrix}Q_1^{-1}Q_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix},$$

and these maps are clearly diffeomorphisms (in fact, real-analytic), and so $M^m_{k,n}(\mathbf{R})$ is a differentiable submanifold of $\mathfrak{M}_{k,n}(\mathbf{R})$. The same procedure can be used to define complex-analytic structures on $\mathfrak{M}_{k,n}(\mathbf{C})$, $M_{k,n}(\mathbf{C})$, and $M^m_{k,n}(\mathbf{C})$, the corresponding sets of matrices over $\mathbf{C}$.

Example 1.8 (Grassmannian manifolds): Let V be a finite dimensional K-vector space and let $G_k(V) := \{\text{the set of } k\text{-dimensional subspaces of } V\}$, for $k < \dim_K V$. Such a $G_k(V)$ is called a *Grassmannian manifold.* We shall use two particular Grassmannian manifolds, namely

$$G_{k,n}(\mathbf{R}) := G_k(\mathbf{R}^n) \qquad \text{and} \qquad G_{k,n}(\mathbf{C}) := G_k(\mathbf{C}^n).$$

The Grassmannian manifolds are clearly generalizations of the projective spaces [in fact, $\mathbf{P}(V) = G_1(V)$; see Example 1.6] and can be given a manifold structure in a fashion analogous to that used for projective spaces.

Consider, for example, $G_{k,n}(\mathbf{R})$. We can define the map

$$\pi \colon M_{k,n}(\mathbf{R}) \longrightarrow G_{k,n}(\mathbf{R}),$$

where

$$\pi(A) = \pi\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{pmatrix} := \{k\text{-dimensional subspace of } \mathbf{R}^n \text{ spanned by the row vectors } \{a_j\} \text{ of } A\}.$$

We notice that for $g \in GL(k, \mathbf{R})$ (the $k \times k$ nonsingular matrices) we have $\pi(gA) = \pi(A)$ (where gA is matrix multiplication), since the action of g merely changes the basis of $\pi(A)$. This is completely analogous to the projection $\pi \colon \mathbf{R}^{n+1} - \{0\} \longrightarrow \mathbf{P}_n(\mathbf{R})$, and, using the same reasoning, we see that $G_{k,n}(\mathbf{R})$ is a compact Hausdorff space with the quotient topology and that π is a surjective, continuous open map.†

We can also make $G_{k,n}(\mathbf{R})$ into a differentiable manifold in a way similar to that used for $\mathbf{P}_n(\mathbf{R})$. Consider $A \in M_{k,n}$ and let $\{A_1, \ldots, A_l\}$ be the collection of $k \times k$ minors of A (see Example 1.7). Since A has rank k, A_α is nonsingular for some $1 \le \alpha \le l$ and there is a permutation matrix P_α such that

$$AP_\alpha = [A_\alpha \tilde{A}_\alpha],$$

where $\tilde{A}_\alpha$ is a $k \times (n-k)$ matrix. Note that if $g \in GL(k, \mathbf{R})$, then gA_α is a nonsingular minor of gA and $gA_\alpha = (gA)_\alpha$. Let $U_\alpha = \{S \in G_{k,n}(\mathbf{R}) \colon S = \pi(A)$, where A_α is nonsingular$\}$. This is well defined by the remark above concerning the action of $GL(k, \mathbf{R})$ on $M_{k,n}(\mathbf{R})$. The set U_α is defined by the condition $\det A_\alpha \neq 0$; hence it is an open set in $G_{k,n}(\mathbf{R})$, and $\{U_\alpha\}_{\alpha=1}^{l}$ covers $G_{k,n}(\mathbf{R})$. We define a map

$$h_\alpha \colon U_\alpha \longrightarrow \mathbf{R}^{k(n-k)}$$

by setting

$$h_\alpha(\pi(A)) = A_\alpha^{-1}\tilde{A}_\alpha \in \mathbf{R}^{k(n-k)},$$

where $AP_\alpha = [A_\alpha \tilde{A}_\alpha]$. Again this is well defined and we leave it to the reader to show that this does, indeed, define a differentiable structure on $G_{k,n}(\mathbf{R})$.

†Note that the compact set $\{A \in M_{k,n}(\mathbf{R}) \colon A\,{}^tA = I\}$ is analogous to the unit sphere in the case $k = 1$ and is mapped surjectively onto $G_{k,n}(R)$.

Example 1.9 (Algebraic submanifolds): Consider $\mathbf{P}_n = \mathbf{P}_n(\mathbf{C})$, and let

$$H = \{[z_0, \ldots, z_n] \in \mathbf{P}_n : a_0 z_0 + \cdots + a_n z_n = 0\},$$

where $(a_0, \ldots, a_n) \in \mathbf{C}^{n+1} - \{0\}$. Then H is called a *projective hyperplane*. We shall see that H is a submanifold of $\mathbf{P}_n$ of dimension $n - 1$. Let U_α be the coordinate systems for $\mathbf{P}_n$ as defined in Example 1.6. Let us consider $U_0 \cap H$, and let $(\zeta_1, \ldots, \zeta_n)$ be coordinates in $\mathbf{C}^n$. Suppose that $[z_0, \ldots, z_n] \in H \cap U_0$; then, since $z_0 \neq 0$, we have

$$a_1 \frac{z_1}{z_0} + \cdots + a_n \frac{z_n}{z_0} = -a_0,$$

which implies that if $\zeta = (\zeta_1, \ldots, \zeta_n) = h_0([z_0, \ldots, z_n])$, then ζ satisfies

$$a_1\zeta_1 + \cdots + a_n\zeta_n = -a_0, \tag{1.2}$$

which is an affine linear subspace of $\mathbf{C}^n$, provided that at least one of $a_1, \ldots, a_n$ is not zero. If, however, $a_0 \neq 0$ and $a_1 = \cdots = a_n = 0$, then it is clear that there is no point $(\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$ which satisfies (1.2), and hence in this case $U_0 \cap H = \varnothing$ (however, H will then necessarily intersect all the other coordinate systems $U_1, \ldots, U_n$). It now follows easily that H is a submanifold of dimension $n - 1$ of $\mathbf{P}_n$ (using equations similar to (1.2) in the other coordinate systems as a representation for H). More generally, one can consider

$$V = \{[z_0, \ldots, z_n] \in \mathbf{P}_n(\mathbf{C}) : p_1(z_0, \ldots, z_n) = \cdots = p_r(z_0, \ldots, z_n) = 0\},$$

where $p_1, \ldots, p_r$ are homogeneous polynomials of varying degrees. In local coordinates, one can find equations of the form (for instance, in U_0)

$$\begin{aligned} p_1\left(1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) &= 0 \\ p_r\left(1, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) &= 0, \end{aligned} \tag{1.3}$$

and V will be a *submanifold* of $\mathbf{P}_n$ if the Jacobian matrix of these equations in the various coordinate systems has maximal rank. More generally, V is called a *projective algebraic variety*, and points where the Jacobian has less than maximal rank are called *singular points* of the variety.

We say that an $\mathcal{S}$-morphism

$$f: (M, \mathcal{S}_M) \longrightarrow (N, \mathcal{S}_N)$$

of two $\mathcal{S}$-manifolds is an $\mathcal{S}$-*embedding* if f is an $\mathcal{S}$-isomorphism onto an $\mathcal{S}$-submanifold of $(N, \mathcal{S}_N)$. Thus, in particular, we have the concept of differentiable, real-analytic, and holomorphic embeddings. Embeddings are most often used (or conceived of as) embeddings of an "abstract" manifold as a submanifold of some more concrete (or more elementary) manifold. Most common is the concept of embedding in Euclidean space and in projective space, which are the simplest geometric models (noncompact and compact, respectively). We shall state some results along this line to give the reader some feeling for the differences among the three categories we have been dealing with. Until now they have behaved very similarly.