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NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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Doina Cioranescu
Jacques-Louis Lions
Editors

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To the memory of Jacques-Louis Lions

Preface

This volume is the 14^{th} and last one of the series "Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar", which published the texts of the lectures given at the seminars organized by Jacques-Louis Lions, from his election at the Collège de France in 1973 until his retirement in 1998. It was one of the foremost seminars in nonlinear PDE's and their applications during that period.

It is unfortunate that because of his untimely death, on May 17, 2001, Jacques-Louis Lions will not see its publication. This volume is dedicated to his memory.

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Chapter 1

AN INTRODUCTION TO CRITICAL POINTS FOR INTEGRAL FUNCTIONALS

D. ARCOYA AND L. BOCCARDO

1. Introduction

The study of minima of functionals defined in spaces of functions may be considered one of the keystones of the mathematical analysis. Remind the efforts by the great mathematicians of the last and present century to develop sufficient conditions on the functional for the existence of minimum. This theory is deeply related with the existence of solutions of boundary value problems. Indeed, this connection is estabilished by the well-known Euler-Lagrange equations associated to the functional.

However, there exist boundary value problems for which the associated functional is indefinite, i.e. it is unbounded from below and from above. This means that it has not global extrema and so we have to search the solutions of the problem among the *critical points*, i.e. the points for which the derivative of the functional vanishes.

From the abstract point of view there is a difference between the study of minima and of critical points. Indeed, for the existence of minima we need only assumptions on the functional. On the contrary, we point out that the results of existence of critical points involve additional hypotheses on the regularity of the functional to assure the existence of a *derivative* in some sense. This may explain why the theory of mimima handles classes of functionals with more general hypotheses of smoothness than the critical point theory.

In some papers [4], [5], [6], we overcame this difference by developing a critical point theory for nondifferentiable functionals. We observe explicitely that our model functionals does not involve similar functions to the modulus. In fact, the nondifferentiability of the considered functionals is due to the introduction of some smooth Carathéodory function $A(x,u)$ (as smooth as you want). Specifically, we consider here functionals J defined in $W_0^{1,2}(\Omega)$

($\Omega \subset \mathbb{R}^N$ open, $N > 1$) by

$$J(v) = \int_\Omega A(x,v)|\nabla v|^2\,dx - \int_\Omega F(x,v^+)\,dx, \quad v \in W_0^{1,2}(\Omega), \qquad (1)$$

with $0 < \alpha \leq A(x,z) \leq \beta < \infty$, $|A'_z(x,z)| \leq \gamma$ and $f(x,z) \equiv F'(x,z)$ (derivative respect to z) a subcritical Carathéodory function. Observe that J is only differentiable along directions of $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, even for smooth functions A (see [11]).

This note is devoted to present the critical point theory developed in [5] (see also [?, ?, ?, ?, ?, ?, ?]) for functionals which are not differentiable in all directions.

2. A mountain pass theorem for nondifferentiable functionals

Our abstract setting for the functionals J that we study is given by the following asumption:

(H) $(X, \|\cdot\|_X)$ is a Banach space and $Y \subset X$ is a subspace which is a normed space endowed with a norm $\|\cdot\|_Y$. Moreover, $J : X \longrightarrow \mathbb{R}$ is a functional on X such that it is continuous in $(Y, \|\cdot\|_X + \|\cdot\|_Y)$ and satisfies the following hypotheses:

a) J has a directional derivative $\langle J'(u), v\rangle$ at each $u \in X$ through any direction $v \in Y$.

b) For fixed $u \in X$, the function $\langle J'(u), v\rangle$ is linear in $v \in Y$, and for fixed $v \in Y$, the function $\langle J'(u), v\rangle$ is continuous in $u \in X$.

Due to the lack of regularity of the functional, some words are needed in order to establish our definition of critical points.

Definition 2.1 – *A function $u \in X$ is called a critical point of J if*

$$\langle J'(u), v\rangle = 0, \quad \forall v \in Y.$$

In this framework a suitable version of the Ambrosetti-Rabinowitz Theorem has been proved in [5]. Specifically,

Theorem 2.2 – *Assume (H) and that for $e \in Y$,*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) > c_1 = \max\{J(0), J(e)\}$$

with Γ the set of the continuous paths $\gamma : [0,1] \longrightarrow (Y, \|\cdot\|_X + \|\cdot\|_Y)$, such that $\gamma(0) = 0$ and $\gamma(1) = e$. Suppose, in addition, that J satisfies the condition

(*C*) *Any sequence* $\{u_n\}$ *in the Banach space* Y *satisfying for some* $\{K_n\} \subset \mathbb{R}^+$ *and* $\{\varepsilon_n\} \longrightarrow 0$ *the conditions*

$$\{J(u_n)\} \text{ is bounded,} \tag{2}$$

$$\|u_n\|_Y \leq 2K_n \quad \forall n \in \mathbb{N}, \tag{3}$$

$$|\langle J'(u_n), v\rangle| \leq \varepsilon_n \left[\frac{\|v\|_Y}{K_n} + \|v\|_X\right] \quad \forall v \in Y, \tag{4}$$

possesses a convergent subsequence in X.

Then there exists a nonzero critical point $u \in Y$ *of* J *such that* $J(u) = c$.

Remarks 2.3.

1. The proof of this theorem is done by dividing it into two steps. In the first one, only the geometric hypotheses are used to deduce the existence of a sequence $\{u_n\}$ in Y satisfying for some $\{K_n\} \subset \mathbb{R}^+$ and $\{\varepsilon_n\} \to 0$ the conditions (2)–(4). The proof is then concluded by using condition (C).

2. In this way, condition (C) can be considered as a compactness condition on the functional J, which substitutes in the nondifferentiable case the role done by the well-known Palais-Smale condition in the regular case $Y = X$.

3. This compactness condition is connected with the coercivity of J extending the previous results for C^1 functionals in [11]. To be specifical in [7] we prove

Theorem 2.4 – *In addition to* (H), *assume that* Y *is dense in* X *and that* J *is continuous in* X *and bounded from below. If* J *satisfies condition* (C) *then* J *is coercive, i.e.,*

$$\lim_{\|u\|_X \longrightarrow \infty} J(u) = \infty.$$

3. A simple model

The application of the abstract result quoted in the previous section to the study of the functional J defined in (1) is very technical. In particular, the verification of the condition (C). For this reason, we present here a simple but not natural functional which is not differentiable in $W_0^{1,2}(\Omega)$, but for which the verification of condition (C) has not technical difficulties like for functionals studied in [5], [6]. Specifically, we consider the functional J defined in $W_0^{1,2}(\Omega)$ by setting

$$\left.\begin{array}{c} J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{q}\int_\Omega a(x,v)|\nabla v|^q\,dx - \frac{1}{m}\int_\Omega (v^+)^m\,dx, \\ v \in W_0^{1,2}(\Omega), \end{array}\right\} \quad (5)$$

where $1 \le q < 2 < m < 2^*$ ($2^* = 2N/(N-2)$, if $2 < N$; $2^* = \infty$ if $N \le 2$) and $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function which is measurable respect to $x \in \Omega$, C^1 with respect to $z \ne 0$ and such that

(a_1) There exist $\beta > \alpha > 0$ such that

$$\alpha \le a(x,z) \le \beta,$$

for almost every $x \in \Omega$ and $z \in \mathbb{R}$.

(a_2) There exists $\gamma > 0$ such that

$$|a'_z(x,z)| \le \gamma, \text{ for almost every } x \in \Omega, \forall z > 0,$$

and $a'(x,z) = 0,\ z < 0$.

(a_3) Either

$$a(x,z) \text{ is increasing and concave with respect to } z \ge 0, \quad (6)$$

or

$$a(x,z) \text{ is decreasing and convex with respect to } z \ge 0. \quad (7)$$

Let $X = W_0^{1,2}(\Omega)$, endowed with the usual norm $\|\cdot\|$; $Y = W_0^{1,2}(\Omega) \cap L^{2/(2-q)}(\Omega)$, endowed with the norm $\|\cdot\|_Y = \|\cdot\|_{2/(2-q)}$. By ($a_1$) and ($a_2$) the functional J is continuous on X and satisfies (H). We point out that $X = Y$ only for $q \le 1 + \frac{2}{N}$.

Theorem 3.1 [5] – *Assume* ($a_1 - a_3$) *and* $1 \le q < 2 < m < 2^*$. *Then the functional* J *defined in* (5) *satisfies* (C).

Proof. Let $\{u_n\}$ be a sequence in Y satisfying (2), (3) and (4) for some $\{K_n\} \subset \mathbb{R}^+$ and let $\{\varepsilon_n\} \longrightarrow 0$. We prove that $\{u_n\}$ is bounded in X. Indeed, taking $v = u_n$ as test function in (4), multiplying (2) by m and adding, we obtain

$$\left(\frac{m}{2}-1\right)\int_\Omega |\nabla u_n|^2\,dx - \frac{1}{q}\int_\Omega a'_z(x,u_n)u_n|\nabla u_n|^q\,dx$$
$$+\left(\frac{m}{q}-1\right)\int_\Omega a(x,u_n)|\nabla u_n|^q\,dx \;\le\; C_1+\varepsilon_n(2+\|u_n\|).$$

Hence, if $a(x,z)$ satisfies (6), taking into account (a_1), it follows for some $C_2>0$ that

$$\left(\frac{m}{q}-1\right)a(x,z)-\frac{1}{q}a'_z(x,z) \;=\; \frac{1}{q}\left[a(x,z)-a'_z(x,z)z\right]$$
$$+\left(\frac{m}{q}-1-\frac{1}{q}\right)a(x,z)\ge -C_2,$$

and thus we deduce

$$\left(\frac{m}{2}-1\right)\int_\Omega |\nabla u_n|^2\,dx \le C_2\int_\Omega |\nabla u_n|^q\,dx + C_1+\varepsilon_n(2+\|u_n\|),$$

which implies, since $q<2$, that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$.

On the other hand, if instead of (6), it holds (7), then that the sequence $\{u_n\}$ is bounded follows easily from the fact that $a'_z(x,z)z\le 0$. Therefore the sequence $\{u_n\}$ is bounded in $X=W_0^{1,2}(\Omega)$. Then there exist $u\in X$ and a subsequence (still denoted u_n) such that u_n converges weakly to u.

Now let

$$T_k(z)=\begin{cases} z, & \text{if } |z|\le k\\[2mm] k\dfrac{z}{|z|}, & \text{if } |z|>k.\end{cases}$$

and

$$G_k(z)=z-T_k(z),\ \forall z\in\mathbb{R},\ \ \forall k>0.$$

To conclude the proof, it suffices to prove

Step 1. $\{T_k(u_n)\}\longrightarrow T_k(u)$ in $W_0^{1,2}(\Omega)$, as $n\to\infty$, for every $k\ge R_1$.

Step 2. For every $\delta>0$, there exist $k_0\ge R_1$ and $n_0\in\mathbb{N}$ such that $\|G_k(u_n)\|<\delta$ for every $k\ge k_0$ and $n\ge n_0$.

Indeed, Steps 1 and 2 imply that, given $\delta>0$, there exist $n_1\in\mathbb{N}$ and $k_1\ge R_1$ such that

$$\begin{aligned}\|u_n - u\| &\le \|u_n - T_{k_1}(u)\| + \|T_{k_1}(u) - u\| \\ &\le \|T_{k_1}(u_n) - T_{k_1}(u)\| + \|G_{k_1}(u_n)\| + \|T_{k_1}(u) - u\| \\ &\le 3\delta, \ \forall n \ge n_1,\end{aligned}$$

i.e. $\{u_n\}$ is strongly convergent in $W_0^{1,2}(\Omega)$ to $u \in W_0^{1,2}(\Omega)$.
Step 1. Putting $w_{n,k} = T_k(u_n) - T_k(u)$ as test function in (4), we deduce

$$\begin{aligned}\int_\Omega \nabla u_n \cdot \nabla w_{n,k}\, dx &+ \int_\Omega a(x,u_n)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla w_{n,k}\, dx \\ &+ \frac{1}{q}\int_\Omega a'_z(x,u_n) w_{n,k}|\nabla u_n|^q\, dx \le \varepsilon'_n\end{aligned}$$

with $\{\varepsilon'_n\} \longrightarrow \infty$. Remark that

$$\begin{aligned}\int_\Omega a(x,u_n)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla w_{n,k} dx \ge \int_\Omega a(x,u_n)|\nabla T_k(u)|^{q-2}\nabla T_k(u)\cdot\nabla w_{n,k} dx \\ + \int_{|u_n|>k} a(x,u_n)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla T_k(u) dx\end{aligned}$$

and the right hand side converges to zero.

Moreover,

$$\int_\Omega a'_z(x,u_n) w_{n,k}|\nabla u_n|^q dx \le C_1\left(\int_\Omega |\nabla u_n|^2 \nabla u_n dx\right)^{q/2} \|T_k(u_n) - T_k(u)\|_{q/(q-2)}$$

Thus, it follows that the sequence $T_k(u_n)$ is convergent in $W_0^{1,2}(\Omega)$ to $T_k(u)$ for every $k > 0$.

Step 2. The assertion is easily proved by taking $G_k(u_n)$ as test function in (4) and using (a_3). ■

Thanks to the previous lemma, we can prove existence of a nontrivial critical point for the functional J. That is, the existence of a weak solution of the quasilinear Dirichlet problem

$$\left.\begin{aligned}-\Delta u - \text{div}\ \left(a(x,u)|\nabla u|^{q-2}\nabla u\right) + \tfrac{1}{q} a'_z(x,u)|\nabla u|^q = |u|^{m-2}u \\ u \in W_0^{1,2}(\Omega),\ u \ge 0 \text{ in } \Omega\end{aligned}\right\}$$

Theorem 3.2 – *Assume $(a_1 - a_3)$ and $1 \le q < 2 < m < 2^*$. Then the functional J given by (5) has at least a positive critical point.*

Proof. We point out that every nonzero critical point of J is positive. In fact, it is deduced taking $T_k(u^-)$ as test function (note that u may not belong to Y, but $T_k(u^-) \in Y$). In order to show the existence of a nonzero critical point, and following the ideas of Lemma 3.3 in [2], it is easy to check that $u = 0$ is a strict local minimum of J, that is, there exist ρ, $\bar{R}$ such that

$$J(u) \geq \rho > 0 \qquad \text{for } \|u\| = \bar{R} > 0. \tag{8}$$

Moreover, $\lim_{|t|\to\infty} J(t\varphi_1) = -\infty$, being $\varphi_1 > 0$ an eigenfunction associated to the first eigenvalue λ_1 of the homogeneous Dirichlet problem for the laplacian operator with L^2-norm equal to one. Thus, there exists $t_0 > \bar{R}$ such that $J(t_0\varphi_1) < 0$. Thus, letting $e = t_0\varphi_1$ and considering the set Γ of the (continuous) paths $\gamma : [0,1] \longrightarrow \left(W_0^{1,2}(\Omega) \cap L^{2/(2-q)}(\Omega), \|\cdot\| + \|\cdot\|_{2/(2-q)}\right)$ which join 0 and e, i.e. such that $\gamma(0) = 0$ and $\gamma(1) = t_0\varphi_1$, we observe that every $\gamma \in \Gamma$ is continuous from $[0,1]$ to $W_0^{1,2}(\Omega)$, so that, by (8), for every $\gamma \in \Gamma$ there exists $\bar{t} \in [0,1]$ such that

$$\|\gamma(\bar{t})\| = \bar{R}.$$

Hence

$$c \equiv \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \geq \rho > \max\{J(0), J(t_0\varphi_1)\} = 0.$$

Then, taking into account Lemma 3.1 and applying Theorem 2.2, we deduce the existence of a critical point $u \in W_0^{1,2}(\Omega) \cap L^{2/(2-q)}(\Omega)$, of J with $J(u) = c > 0$ and thus $u \neq 0$. ■

4. Main examples

The abstract theorem (with $X = W_0^{1,2}(\Omega)$ and $Y = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$) of the Section 2 is applied now to obtain nonnegative critical points of the functional $J : W_0^{1,2}(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$J(v) = \int_\Omega A(x,v)|\nabla v|^2\,dx - \int_\Omega F(x,v^+)\,dx, \;\; v \in W_0^{1,2}(\Omega), \tag{9}$$

i.e. nonnegative solutions of the boundary value problem:

$$\left.\begin{aligned} &u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}(\,A(x,u)\,\nabla u) + &\tfrac{1}{2}A'_z(x,u)|\nabla u|^2 = F'_u(x,u) \equiv f(x,u) \end{aligned}\right\} \tag{P}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function with subcritical growth. It is clear that for a solution u of (P) we are meaning

$$\left.\begin{array}{c} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \\ \int_\Omega A(x,u)\nabla u \nabla v dx + \frac{1}{2}\int_\Omega A'_z(x,u)|\nabla u|^2 v dx = \int_\Omega f(x,u) v dx \end{array}\right\}$$

for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

The hypotheses that we assume on the Carathéodory coefficient $A : \Omega \times \mathbb{R} \to \mathbb{R}$ are the following:

(A_1) There exists $\alpha > 0$ such that

$$\alpha \leq A(x,z),$$

for almost every $x \in \Omega$ and $z \geq 0$.

(A_2) There exists $R_1 > 0$ such that $A'_z(x,z) \geq 0$ for almost every $x \in \Omega$, for every $z \geq R_1$.

(A_3) There exist $m > 2$ and $\alpha_1 > 0$ such that

$$\left(\frac{m-2}{2}\right) A(x,z) - \frac{1}{2} z A'_z(x,z) \geq \alpha_1,$$

for almost every $x \in \Omega$, $z \geq 0$.

Notice that all assumptions on $A(x,z)$ are for $z \geq 0$. In fact, since we are looking for nonnegative solutions of (P) we can assume without loss of generality that $A(x,z)$ is even on z.

On the other hand, we will assume the following conditions on $f(x,z)$:

(f_1) There exist $C_1, C_2 > 0$ such that

$$|f(x,z)| \leq C_1|z|^\sigma + C_2, \quad \text{a.e. } x \in \Omega, \quad \forall z \in \mathbb{R}^+,$$

with $\sigma + 1 < 2^*$, ($2^* = 2N/(N-2)$ if $2 < N$, and $2^* = \infty$ if $2 \geq N$).

(f_2) There exists $R_2 > 0$ such that

$$0 < mF(x,z) \leq z f(x,z),$$

for almost $x \in \Omega$ and every $z \geq R_2$ (m is the same as in (A_3)).

(f_3) $f(x,|z|) = o(|z|)$ at $z = 0$, uniformly in $x \in \Omega$.

Theorem 4.1 – *Assume (A_{1-3}), (f_{1-3}) and that*

$$\lim_{z\to+\infty} \frac{A(x,z)}{z^\sigma} = 0, \quad \textit{unif. in } x \in \Omega. \tag{10}$$

Then the problem (P) has, at least, one nonnegative and nontrivial solution.

Remarks 4.2.

1. The above theorem is essentially contained in [5]. However, in that paper it is assumed in addition that $A(x,z)$ is bounded from above and its derivative $A'_z(x,z)$ with respect to z is also bounded. In [7], we have seen that these additional hypotheses are not necessary for the existence.

2. The general case of functionals

$$\int_\Omega \mathcal{J}(x,v,\nabla v)\,dx - \int_\Omega F(x,v^+)\,dx, \quad v \in W_0^{1,p}(\Omega), \ (p>1)$$

could be also handled as in [5]. For simplicity reasons, we just present here the case $p=2$, $\mathcal{J}(x,v,\nabla v) = A(x,v)|\nabla v|^2$.

3. Some remarks about the meaning of (A_3) and (f_2) may be found in [5, Lemma 3.2 and Remarks 3.1]. ■

Proof of Theorem 4.1. For every $n \in \mathbb{N}$, let h_n be a nondecreasing C^1 function in $[0,\infty)$ satisfying

$$h_n(s) = s, \ \forall s \in [0, n-1],$$

$$h_n(s) \le s, \ \forall s \in (n-1, n),$$

$$h_n(s) = n, \ \forall s \ge n.$$

Consider the coefficients $A_n(x,z) \equiv h_n(A(x,z))$, $x \in \Omega$, $z \in \mathbb{R}$. Clearly, A_n satisfies (A_{1-3}) and, in addition, it is bounded from above with bounded derivative $A'_n(x,z)$ (with respect to z). In this way, if we define the functionals $J_n : W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}$ by setting

$$J_n(v) = \int_\Omega A_n(x,v)|\nabla v|^2\,dx - \frac{1}{s+1}\int_\Omega F\left(x,v^+\right)\,dx, \quad v \in W_0^{1,2}(\Omega),$$

then using (f_{1-2}) and (A_3), it can be seen in a similar way to the one in Section 2 that J_n satisfies (C). Indeed, we have

Lemma 4.3 – *(Compactness condition) Assume* (A_{1-3}) *and* (f_{1-2})*. Then the functional* J_n *satisfies* (C)*.*

Using in addition (f_3) and following the ideas of [2], it is easily seen that J_n satisfies the geometrical hypotheses of Theorem 2.2. Consequently, by it, there exists a nontrivial and nonnegative solution u_n of the problem

$$\left.\begin{array}{c} u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}(\, A_n(x, u_n) \nabla u_n) + \frac{1}{2} A_n'(x, u_n)|\nabla u_n|^2 = f(x, u_n^+). \end{array}\right\} \quad (11)$$

with critical level

$$J_n(u_n) = c_n \equiv \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_n(\gamma(t)),$$

where $\Gamma = \{\gamma : [0,1] \longrightarrow W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \,/\, \gamma(0) = 0, \gamma(1) = e_n\}$, $e_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is such that $J_n(e_n) < 0$. Taking into account that $A_n(x,z) \le A(x,z)$ and (10), we observe that

$$J_n(t\varphi_1) \le J(t\varphi_1) = \frac{1}{2} t^{s+1} \left[\in_\Omega \frac{A(x, t\varphi_1)}{t^{s-1}} |\nabla \varphi_1|^2 \, dx - \frac{1}{s+1} \int_\Omega \varphi_1^{s+1} \, dx \right] < 0,$$

for all $t \in [t_0, \infty)$ if $t_0 > 0$ is large enough. This allows us to choose $e_n = t_0 \varphi_1$ (independent of $n \in \mathbb{N}$). On the other hand, by the Mountain Pass geometry of J_1 there exist $\delta, r > 0$ such that

$$J_n(v) \ge J_1(v) \ge \delta, \ \forall \|v\| \le r,$$

(i.e., roughly speaking, $v = 0$ is a strict local minimum of J_n, uniformly in $n \in \mathbb{N}$). This implies that

$$J_n(u_n) = c_n \ge \delta. \quad (12)$$

We claim that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Indeed, using again that $A_n(x,z) \le A(x,z)$, we deduce

$$\begin{aligned} J_n(u_n) &= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_n(\gamma(t)) \\ &\le \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \\ &\le \max_{t \in [0,1]} J(t t_0 \varphi_1) \equiv C_1. \end{aligned}$$

Subtracting $\frac{1}{m} \langle J_n'(u_n), u_n \rangle = 0$ we derive