Elias M. Stein, Guido Weiss

Introduction to Fourier Analysis on Euclidean Spaces

欧几里得空间的傅里叶分析

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By Elias M. Stein
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THIS BOOK IS DEDICATED TO

Antoni Zygmund

IN APPRECIATION FOR HIS FRIENDSHIP,
HIS TEACHING, AND THE INSPIRATION
HE GAVE US

Preface

This book is designed to be an introduction to harmonic analysis in Euclidean spaces. The subject has seen a considerable flowering during the past twenty years. We have not tried to cover all phases of this development. Rather, our chief concern was to illustrate various methods used in this aspect of Fourier analysis that exploit the structure of Euclidean spaces. In particular, we try to show the role played by the action of translations, dilations, and rotations. Another concern, not independent of this chief one, is to motivate the study of harmonic analysis on more general spaces having an analogous structure (such as arises in symmetric spaces). It is our feeling that the study of Fourier analysis in that context and, also, in other general settings, is more meaningful once the special Euclidean case is understood.

Because of these concerns we have not included several topics that are usually presented in more general treatments of harmonic analysis. For example, results centering around the Wiener Tauberian theorem, which hold in the case of general locally compact Abelian groups, do not involve the special features of Euclidean spaces. Stated very briefly, our selection of topics was motivated by showing how real variable and complex variable methods extend from the one-dimensional to the many-dimensional case.

We require that the readers have mastered the material that is usually covered in a course in the theory of integration and in the theory of functions of a complex variable. We also believe that this book is much more meaningful to someone who has some previous acquaintance with harmonic analysis. If the reader has no such previous experience, he might find it profitable first to look at the expository article "Harmonic Analysis" (see Guido Weiss [3]).

The idea of writing this book first occurred to us when we presented some of these topics in graduate courses given during the academic year 1958–59. Since that time each of us has lectured on this subject at various times and places. We are thankful to our colleagues and students who helped us clarify our ideas and organize this material.

More specifically, one of us (Weiss) taught a course on this subject at Washington University jointly with Mitchell Taibleson during the academic

year 1963–64. We are grateful for his contribution to the organization of this course and his continued interest in our effort. The other author gave courses at the University of Chicago (1961–62) and Princeton (1963–65) that dealt with the topics of this book. Our task of writing the book was considerably facilitated by having the lecture notes, which were prepared by R. Askey, N. J. Weiss, and D. Levine. It is a pleasure to express our appreciation to them. Ronald R. Coifman was a constant help throughout the writing of this book. His many suggestions are incorporated in much of the presentation. Miguel de Guzmán, J. R. Hattemer, Stephen Wainger, and N. J. Weiss read the entire manuscript. We are grateful for their corrections and comments. We are also indebted to I. I. Hirschman Jr., who read part of the manuscript and made several useful suggestions, and to Mrs. A. Bonami, J. L. Clerc, L. J. Dickson, S. S. Gelbart, and S. Zucker who helped us correct the proof sheets.

Princeton, N.J.
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Introduction to Fourier Analysis
on Euclidean Spaces

Contents

CHAPTER I

The Fourier Transform

In this chapter we introduce the Fourier transform and study its more elementary properties. Since most of the material of this chapter is rather standard our treatment here will be brief. We begin by considering the behavior of the Fourier transform on the spaces $L^1(E_n)$ and $L^2(E_n)$. This will be done in the first two sections. The Fourier transform's formal aspects are more easily described in the context of distributions; therefore, in the third section we extend its definition to the space of tempered distributions. The reader will note that in this chapter we are mainly exploiting the translation structure of Euclidean spaces. In the following chapters (and specifically in Chapter IV), however, the action of rotations on these spaces plays an important role.

1. The Basic L^1 Theory of the Fourier Transform

We begin by introducing some notation that will be used throughout this work. E_n denotes n-dimensional (real) Euclidean space. We consistently write $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n), \ldots$ for the elements of E_n. The *inner product* of $x, y \in E_n$ is the number $x \cdot y = \sum_{j=1}^n x_j y_j$; the *norm* of $x \in E_n$ is the (nonnegative) number $|x| = \sqrt{x \cdot x}$; furthermore, $dx = dx_1\, dx_2 \cdots dx_n$ denotes the element of ordinary Lebesgue measure.

We will deal with various spaces of functions defined on E_n. The simplest of these are the $L^p = L^p(E_n)$ spaces, $1 \leqq p < \infty$, of all measurable functions f such that $\|f\|_p = (\int_{E_n} |f(x)|^p \, dx)^{1/p} < \infty$. The number $\|f\|_p$ is called the *L^p norm* of f. The space $L^\infty(E_n)$ consists of all essentially bounded functions on E_n and, for $f \in L^\infty(E_n)$, we let $\|f\|_\infty$ be the essential supremum of $|f(x)|$, $x \in E_n$.[1] Often, the space C_0 of all continuous functions vanishing at infinity, with the L^∞ norm just described, arises more naturally than $L^\infty = L^\infty(E_n)$. Unless otherwise specified, all functions are assumed to be

[1] We say f is *equivalent* to g if $f(x) = g(x)$ for almost every $x \in E_n$, whenever f $g \in L^p(E_n)$, $1 \leqq p \leqq \infty$. If we consider the equivalence classes obtained from this relation and define the *norm* of a class to be the norm of any one of its representatives (clearly $\|f\|_p = \|g\|_p$ if f is equivalent to g) we obtain a Banach space. We shall denote this space by $L^p(E_n)$ as well. It will be obvious from the context which of these spaces $L^p(E_n)$ is under discussion.

complex valued; it will be assumed, throughout the book, that all functions are (Borel) measurable.

If $f \in L^1(E_n)$ the *Fourier transform* of f is the function $\hat{f}$ defined by letting

$$\hat{f}(x) = \int_{E_n} f(t)e^{-2\pi i x \cdot t}\, dt$$

for all $x \in E_n$. It is easy to establish the following results:

THEOREM 1.1. (a) *The mapping $f \to \hat{f}$ is a bounded linear transformation from $L^1(E_n)$ into $L^\infty(E_n)$. In fact $\|\hat{f}\|_\infty \leqq \|f\|_1$;*

(b) *If $f \in L^1(E_n)$ then $\hat{f}$ is uniformly continuous.*

THEOREM 1.2 (Riemann–Lebesgue). *If $f \in L^1(E_n)$ then $\hat{f}(x) \to 0$ as $|x| \to \infty$; thus, in view of the last result, we can conclude that $\hat{f}$ belongs to the class C_0.*

Theorem 1.1 is obvious; moreover, so is Theorem 1.2 when f is the characteristic function of the n-dimensional interval $I = \{x \in E_n;\, a_1 \leqq x_1 \leqq b_1, \ldots, a_n \leqq x_n \leqq b_n\}$ (for we can calculate $\hat{f}$ explicitly as an iterated integral). The same is therefore true for a finite linear combination of such characteristic functions. The result for a general $f \in L^1(E_n)$ follows easily by approximating f in the L^1 norm by such a linear combination g; for then $f = g + (f - g)$, where $\hat{f} - \hat{g}$ is uniformly small (by Theorem 1.1, part (a)) while $\hat{g}(x) \to 0$ as $|x| \to \infty$.

Theorem 1.2 gives a necessary condition for a function to be a Fourier transform. Belonging to the class C_0, however, is far from being sufficient (see 4.1). There seems to be no simple satisfactory condition characterizing Fourier transforms of functions in $L^1(E_n)$.

The above definition of the Fourier transform extends immediately to finite Borel measures: if μ is such a measure on E_n we define $\hat{\mu}$ by letting

$$\hat{\mu}(x) = \int_{E_n} e^{-2\pi i x \cdot t}\, d\mu(t).$$

Theorem 1.1 is valid for these Fourier transforms if we replace the L^1-norm by the total variation of μ.

In addition to the vector-space operations, $L^1(E_n)$ is endowed with a "multiplication" making this space a Banach algebra. This operation, called *convolution*, is defined in the following way: If f and g belong to $L^1(E_n)$ their convolution $h = f * g$ is the function whose value at $x \in E_n$ is

$$h(x) = \int_{E_n} f(x - y)g(y)\, dy.$$

One can show by an elementary argument that $f(x - y)g(y)$ is a measurable function of the two variables x and y. It then follows immediately from Fubini's theorem on the interchange of the order of integration that $h \in L^1(E_n)$ and $\|h\|_1 \leqq \|f\|_1\|g\|_1$. Furthermore, this operation is commutative and associative. More generally, $h = f * g$ is defined whenever $f \in L^p(E_n)$, $1 \leqq p \leqq \infty$, and $g \in L^1(E_n)$. In fact we have the following result:

THEOREM 1.3. *If $f \in L^p(E_n)$, $1 \leqq p \leqq \infty$, and $g \in L^1(E_n)$ then $h = f * g$ is well defined and belongs to $L^p(E_n)$. Moreover,*

$$\|h\|_p \leqq \|f\|_p\|g\|_1.$$

It is clear that $|h(x)| \leqq \int_{E_n} |f(x - y)|\,|g(y)|\,dy$. Thus the desired result is an easy consequence of Minkowski's integral inequality:

$$\left(\int_{E_n} |h(x)|^p\,dx\right)^{1/p} \leqq \int_{E_n}\left(\int_{E_n} |f(x - y)|^p\,dx\right)^{1/p} |g(y)|\,dy$$
$$= \|f\|_p\|g\|_1.$$

As was the case for the Fourier transform, we can extend this operation to include finite Borel measures: if μ is such a measure on E_n we define $h = f * d\mu$ by letting

$$h(x) = \int_{E_n} f(x - y)\,d\mu(y),$$

for $x \in E_n$ and $f \in L^p$. Theorem 1.3 is valid for these convolutions if we replace the L^1-norm of g by the total variation of μ.

An essential feature of harmonic analysis is the fact that the Fourier transform of the convolution of two functions is the (pointwise) product of their Fourier transforms. More precisely, the following result is an easy consequence of the definitions:

THEOREM 1.4. *If f and g belong to $L^1(E_n)$ then*

$$(f * g)\hat{} = \hat{f}\hat{g}.$$ [2]

Many other important operations of analysis have particularly simple relations with the Fourier transform. For example, if we let τ_h denote *translation* by $h \in E_n$ (by this we mean the operator mapping the function $g(x)$ into the function $g(x - h)$) we have

(1.5)
$$\text{(i)}\quad (\tau_h f)\hat{}(x) = e^{-2\pi i h \cdot x}\hat{f}(x),$$
$$\text{(ii)}\quad (e^{2\pi i t \cdot h}f(t))\hat{}(x) = (\tau_h \hat{f})(x),$$

whenever $f \in L^1(E_n)$.

[2] We shall use this notation consistently: $(\ldots)\hat{}$ denotes the Fourier transform of $(\ldots)$.

If $a > 0$ we let δ_a denote *dilation by* a; that is, δ_a is the operator mapping the function $g(x)$ into the function $g(ax)$. Whenever $f \in L^1(E_n)$ we then have

$$a^n(\delta_a f)\hat{}(x) = \hat{f}(a^{-1}x). \tag{1.6}$$

Differentiation and Fourier transformation are related in the following way:

THEOREM 1.7. *Suppose $f \in L^1(E_n)$ and $x_k f(x) \in L^1(E_n)$, where x_k is the k-th coordinate function. Then $\hat{f}$ is differentiable with respect to x_k and*

$$\frac{\partial \hat{f}}{\partial x_k}(x) = (-2\pi i t_k f(t))\hat{}(x).$$

PROOF. Letting $h = (0, \ldots, h_k, \ldots, 0)$ be a nonzero vector along the k-th coordinate axis, we have, by part (ii) of (1.5) and the Lebesgue dominated convergence theorem,

$$\frac{\hat{f}(x+h) - \hat{f}(x)}{h_k} = \left\{\left(\frac{e^{-2\pi i t\cdot h} - 1}{h_k}\right) f(t)\right\}\hat{}(x) \to (-2\pi i t_k f(t))\hat{}(x)$$

as $h_k \to 0$.

Theorem 1.7 asserts that applying the Fourier transform after multiplying by the k-th coordinate function is equivalent (up to a multiplicative constant) to taking the partial derivative with respect to the k-th variable of the Fourier transform. It is also true that the Fourier transforms of such partial derivatives are obtainable (again, up to a multiplicative constant) by multiplying the Fourier transform by the corresponding coordinate functions. We shall encounter many versions of this result. In order to make a precise statement of one of these versions (perhaps the simplest to prove) we introduce the following concept: We say that f is *differentiable in the L^p norm with respect to x_k* whenever $f \in L^p(E_n)$ and there exists a g in $L^p(E_n)$ such that

$$\left(\int_{E_n} \left|\frac{f(x+h) - f(x)}{h_k} - g(x)\right|^p dx\right)^{1/p} \to 0,$$

as $h_k \to 0$ (we are using the notation established in the proof of Theorem 1.7). The function g is called the *partial derivative of f (with respect to x_k) in the L^p norm.*

Applying part (i) of (1.5) and part (a) of Theorem 1.1 to

$$|\hat{g}(x) - \hat{f}(x)(e^{2\pi i(h\cdot x)} - 1)/h_k|,$$

and then letting $h_k \to 0$ we obtain:

THEOREM 1.8. *If $f \in L^1(E_n)$ and g is the partial derivative of f with respect to x_k in the L^1 norm then*

$$\hat{g}(x) = 2\pi i x_k \hat{f}(x).$$

Both Theorems 1.7 and 1.8 can be extended to higher derivatives. Without going into details, we note the following formulas:

$$\text{(1.9)}\qquad \begin{aligned} &\text{(i)}\quad P(D)\hat{f}(x) = (P(-2\pi it)f(t))\hat{\ }(x),\\ &\text{(ii)}\quad (P(D)f)\hat{\ }(x) = P(2\pi ix)\hat{f}(x),\end{aligned}$$

where, for an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers we let $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $D^\alpha = \partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}$, P is a polynomial in the n variables $x_1, x_2, \ldots, x_n$ and $P(D)$ is the associated differential operator (i.e., we replace x^α by D^α in $P(x)$).

We now turn to the problem of inverting the Fourier transform. That is, we shall consider the question: *Given the Fourier transform $\hat{f}$ of an integrable function f, how do we obtain f back again from $\hat{f}$?* The reader familiar with the elementary theory of Fourier series and integrals would expect $f(t)$ to be equal to the integral

$$\text{(1.10)}\qquad \int_{E_n} \hat{f}(x) e^{2\pi it\cdot x}\, dx.$$

Unfortunately, $\hat{f}$ need not be integrable (for example, let $n = 1$ and f be the characteristic function of a finite interval). In order to get around this difficulty we shall use certain summability methods for integrals. We first introduce the *Abel* method of summability, whose analog for series is very well-known. For each $\varepsilon > 0$ we define the *Abel mean* $A_\varepsilon = A_\varepsilon(f)$ to be the integral

$$A_\varepsilon(f) = A_\varepsilon = \int_{E_n} f(x) e^{-\varepsilon|x|}\, dx.$$

It is clear that if $f \in L^1(E_n)$ then $\lim_{\varepsilon\to 0} A_\varepsilon(f) = \int_{E_n} f(x)\, dx$. On the other hand, these Abel means are well-defined even when f is not integrable (if we only assume, for example, that f is bounded, then $A_\varepsilon(f)$ is defined for all $\varepsilon > 0$). Moreover, their limit

$$\text{(1.11)}\qquad \lim_{\varepsilon\to 0} A_\varepsilon(f) = \lim_{\varepsilon\to 0} \int_{E_n} f(x) e^{-\varepsilon|x|}\, dx$$

may exist even when f is not integrable. A classical example of such a case is obtained by letting $f(x) = \sin x/x$ when $n = 1$.[3]

Whenever the limit in (1.11) exists and is finite we say that $\int_{E_n} f$ is *Abel summable* to this limit.

A somewhat similar method of summability is *Gauss summability*. This method is defined by the *Gauss* (sometimes called *Gauss–Weierstrass*)

[3] As is well known, in this case $\lim_{p\to\infty} \int_0^p f(x)\, dx$ exists. It is an easy exercise to show that whenever f is locally integrable and such a limit, l, exists the Abel means $A_\varepsilon = \int_0^\infty e^{-\varepsilon x} f(x)\, dx$ converge to l.