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C_0-Semigroups and Applications

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To my wife Veronica

Preface

This book is an entirely rewritten English version of the lecture notes of an advanced course I taught during the last eleven years at the Faculty of Mathematics of "Al. I. Cuza" University of Iaşi. Lecture notes appeared in 2001 in Romanian. The idea was to give a unified and systematic presentation of a fundamental branch of operator theory: the linear semigroups. The existence of several very good books on this topic such as : Ahmed [**2**], Belleni-Morante [**24**], Butzer and Berens [**32**], Davies [**45**], Engel and Nagel et al [**51**], Goldstein [**61**], Haraux [**68**], Hille and Phillips [**70**], McBride [**89**], and Pazy [**101**] made this task very hard to accomplish. Nevertheless, I decided to accept it, simply because there are several particular topics which have not found their place into a monograph until now, mainly because they are very new. This book, although containing the main parts of the classical theory of C_0-semigroups, as the Hille-Yosida theory, illustrated by a wealth of applications of both traditional and non-standard mathematical models, also includes some new, or even unpublished results. We refer here to: the characterization in terms of real regular values of both differentiable and analytic semigroups, the study of elliptic and parabolic systems with dynamic boundary conditions, the study of linear and semilinear differential equations with distributed measures, as well as a finite-dimensional like treatment of semilinear hyperbolic equations, mainly due to the author. As far as I know, some other topics appear for the first time in a book form here: the equations of linear thermoelasticity, the equations of linear viscoelasticity and the characterization of generators of equicontinuous and of compact semigroups, being the most important ones. Besides, the last part of the book contains detailed solutions to all the problems included at the end of each chapter.

There are some interesting topics which, although useful, were not discussed in this book. In this respect I would like to mention the spectral mapping theorems and a thorough study of the asymptotic behavior of solutions. Moreover, in order to avoid some slight complications, most of the results in this book refer only to C_0-semigroups of contractions, although they hold true for general C_0-semigroups, i.e. of type (M, ω). However, I assume that the interested readers will be able to fill in this gap, if necessary.

I believe that someone who has some acquaintance with functional analysis and differential equations can read the book. Therefore, I hope that it will be found useful not only by graduate students and researchers in Mathematics to whom it is primarily addressed, but also by physicists and engineers interested in deterministic mathematical models expressed in terms of differential equations.

I am greatly thankful to my former professors and students, as well as to my colleagues and friends who helped me to clarify many ideas and to organize the presentation. More specifically, I am grateful to professor Viorel Barbu for the courses he taught, which had a decisive influence on my further evolution, and for his unceasing interest in my efforts. The discussions with Professor Dorin Ieşan were of great help to me in order to clarify some aspects concerning the examples in Mechanics presented in Sections 4.8, 4.9 and 10.5. The writing of this book was facilitated by a very careful reading of the manuscript followed by many suggestions and comments by Professors Ovidiu Cârjă, Mihai Necula and Constantin Zălinescu, by Dr. Corneliu Ursescu, Senior Researcher at The "Octav Mayer" Institute of Mathematics of the Romanian Academy, as well as by Dr. Silvia-Otilia Corduneanu. Both Professor Cătălin Lefter and my former student Eugen Vărvăruca read the entire Romanian version of the manuscript and made several useful remarks I took into account in the presentation. Dr. Ioana Sîrbu from SUNY at Buffalo was of great help to make the English read smoothly.

It is a great pleasure to express my appreciation to all of them.

Iaşi, November 12^{th}, 2002 Ioan I. Vrabie

Contents

CHAPTER 1

Preliminaries

The aim of this chapter is to give a brief presentation of some auxiliary notions and results which are needed for a good understanding of the whole book. In the first two sections, we define and study the class of vector-valued measurable functions as well as the integral of such functions with respect to a σ-finite and complete measure. In the third section, we recall the definition of the spaces $\mathcal{L}^p(\Omega, \mu; X)$ and $L^p(\Omega, \mu; X)$, with (Ω, Σ, μ) a σ-finite and complete measure space, and X a Banach space, and we recall their most remarkable properties. Also here, we present some properties of $W^{k,p}(a, b\,; X)$ and $A^{k,p}(a, b\,; X)$. The fourth section is devoted to a short presentation of the space $BV([\,a, b\,]; X)$ of functions of bounded variation from $[\,a, b\,]$ to X, while in the fifth section, we collect several results referring to Sobolev spaces, exactly in form they will be used later in the book. The sixth section contains some basic facts concerning unbounded linear operators in Banach spaces, with main emphasis on self-adjoint and respectively skew-adjoint operators acting in Hilbert spaces. In the seventh section, we include several spectral analysis results with regards to unbounded, closed linear operators on Banach spaces, while in the last two sections, we introduce and study the Dunford integral in order to offer an elegant way to define the value of an analytic scalar function at such an operator.

1.1. Vector-Valued Measurable Functions

Let X be a real Banach space and (Ω, Σ, μ) a σ-finite and complete measure space. We recall that $\mu : \Sigma \to \mathbb{R}_+$ is *σ-finite* if there exists a family $\{\Omega_n;\, n \in \mathbb{N}\} \subset \Sigma$ such that $\mu(\Omega_n) < +\infty$ for each $n \in \mathbb{N}$ and $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$. The measure μ is called *complete* if each subset of a null μ-measure set is measurable (belongs to Σ).

Definition 1.1.1. A function $x : \Omega \to X$ is called:

(i) *countably-valued* if there exist two families: $\{\Omega_n;\, n \in \mathbb{N}\} \subset \Sigma$ and $\{x_n;\, n \in \mathbb{N}\} \subset X$, with $\Omega_k \cap \Omega_p = \emptyset$ for each $k \neq p$, $\Omega = \cup_{n\geq 0}\Omega_n$, and such that $x(\omega) = x_n$ for all $\omega \in \Omega_n$;

(ii) *almost separably-valued* if there exists a μ-null set Ω_0 such that $x(\Omega \setminus \Omega_0)$ is separable;

(iii) *strongly measurable* if there exists a sequence of countably-valued functions convergent to x μ-a.e. on Ω;

(iv) *weakly measurable* if, for each $x^* \in X^*$, the function $x^*(x) : \Omega \to \mathbb{R}$ is measurable[1].

Definition 1.1.2. A subset Λ in X^* is called *determining set* for X if for each $x \in X$ we have

$$\|x\| = \sup\{|x^*(x)|;\ x^* \in \Lambda\}.$$

Remark 1.1.1. If Λ is a determining set for X then its elements have the norm at most equal to 1. This is a consequence of the definition of the usual sup-norm on X^*.

Theorem 1.1.1. *Each separable Banach space has at least one countable determining set.*

Proof. Let $\{x_n;\ n \in \mathbb{N}\}$ a dense subset in X. Since for each $x \in X$ we have

$$\|x\| = \sup\{|x^*(x)|;\ x^* \in X^*,\ \|x^*\| = 1\},$$

it follows that there exists a family $\{x^*_{n,m};\ n, m \in \mathbb{N}\}$ in the unit closed ball in X^*, such that, for each $n \in \mathbb{N}$,

$$\lim_{m\to\infty} |x^*_{n,m}(x_n)| = \|x_n\|.$$

As $\|x^*_{n,m}\| = 1$ for $n, m \in \mathbb{N}$, $\|x_n\| = \sup\{|x^*_{n,m}(x_n)|;\ m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$ and $\{x_n;\ n \in \mathbb{N}\}$ is dense in X, we deduce that

$$\|x\| = \sup\{|x^*_{n,m}(x)|;\ n, m \in \mathbb{N}\}$$

for each $x \in X$. The proof is complete. □

Theorem 1.1.2. *If X admits a countable determining set Λ and $x : \Omega \to X$ is weakly measurable, then $\|x\| : \Omega \to \mathbb{R}_+$ is measurable.*

Proof. Since the supremum of a countable family of real measurable functions is a measurable function and

$$\|x(\omega)\| = \sup\{|x^*(x(\omega))|;\ x^* \in \Lambda\}$$

for each $\omega \in \Omega$, where each function $x^*(x)$ is measurable, it follows that $\|x\|$ has the same property and this achieves the proof. □

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[1]Some authors prefer the term *scalarly measurable* instead of weakly measurable, keeping the latter term for those functions $x : \Omega \to X$ with the property that, for each weakly open subset D in X, $x^{-1}(D) \in \Sigma$.

Theorem 1.1.3. (Pettis) *A function $x : \Omega \to X$ is strongly measurable if and only if it is weakly measurable and almost separably-valued.*

Proof. *Necessity.* As x is strongly measurable there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of countably-valued functions and a μ-null set Ω_0, such that

$$\lim_{n\to\infty} x_n(\omega) = x(\omega) \tag{1.1.1}$$

for each $\omega \in \Omega\setminus\Omega_0$. But each function in the sequence is at most countably-valued, and thus $\cup_{n\geq 0}\{x_n(\omega);\ \omega \in \Omega\}$ is at most countable and dense in $x(\Omega\setminus\Omega_0)$. Hence x is almost separably-valued.
From (1.1.1) we conclude that, for each $x^* \in X^*$ and $\omega \in \Omega\setminus\Omega_0$, we have

$$\lim_{n\to\infty} x^*(x_n(\omega)) = x^*(x(\omega)).$$

Taking into account that the functions $x^*(x_n)$ are almost countably-valued, and thus measurable, it follows that $x^*(x)$ is measurable.
Sufficiency. Since x is almost separably-valued, we may assume with no loss of generality that X is separable. Indeed, if X is not separable, let us consider the μ-null set Ω_0 such that $x(\Omega\setminus\Omega_0)$ is separable and let Y be the closed linear subspace spanned by $x(\Omega\setminus\Omega_0)$. Obviously this is separable and, in addition, x coincides μ-a.e. with a function y defined on Ω and taking values in Y. It is easy to see that y is strongly measurable if and only if x enjoys the same property. Similarly, x is weakly measurable if and only if y is weakly measurable, since, by virtue of the Hahn-Banach theorem (see Theorem 2.7.1, p. 29 in Hille and Phillips [**70**]), each linear bounded functional on Y coincides with the restriction of a linear bounded functional on X.
So, let $\{x_n;\ n \in \mathbb{N}^*\}$ be a dense subset in X and let $\varepsilon > 0$. We define

$$\Omega_+ = \{\omega \in \Omega;\ x(\omega) \neq 0\} \quad\text{and}\quad \Omega_n^\varepsilon = \{\omega \in \Omega_+;\ \|x(\omega) - x_n\| \leq \varepsilon\}$$

for $n \in \mathbb{N}^*$. From Theorems 1.1.1 and 1.1.2, it follows that both Ω_+ and Ω_n^ε are measurable. Since $\{x_n;\ n \in \mathbb{N}^*\}$ is dense in X, we deduce that, for each $\varepsilon > 0$,

$$\bigcup_{n\geq 1} \Omega_n^\varepsilon = \Omega_+. \tag{1.1.2}$$

Indeed, if we assume by contradiction that this is not the case, then there exist $\varepsilon > 0$ and $\omega \in \Omega_+$ such that $\|x(\omega) - x_n\| > \varepsilon$ for each $n \in \mathbb{N}^*$. But the inequalities above show that $x(\omega)$ does not belong to the closure of the set $\{x_n;\ n \in \mathbb{N}^*\}$ which coincides with X. This contradiction can be eliminated only if (1.1.2) holds. Let us define now

$$E_1^\varepsilon = \Omega_1^\varepsilon \text{ and } E_n^\varepsilon = \Omega_n^\varepsilon \setminus \bigcup_{k=1}^{n-1} \Omega_k^\varepsilon \text{ for } n = 2, 3, \ldots$$

and let us observe that all the sets E_n^ε are measurable and

$$\bigcup_{n\geq 1} E_n^\varepsilon = \Omega_+ \text{ and } E_k^\varepsilon \cap E_p^\varepsilon = \emptyset \text{ for } k \neq p.$$

Let $x_\varepsilon : \Omega \to X$ be defined by

$$x_\varepsilon(\omega) = \begin{cases} x_n & \text{if } \omega \in E_n^\varepsilon \\ 0 & \text{if } \omega \in \Omega \setminus \Omega_+. \end{cases}$$

Obviously x_ε is countably-valued and $\|x(\omega) - x_\varepsilon(\omega)\| \leq \varepsilon$ for each $\omega \in \Omega$. The proof is complete. □

Remark 1.1.2. The definition of x_ε in the proof of Theorem 1.1.3 shows that a function $x : \Omega \to X$ is strongly measurable if and only if there exists a sequence of countably-valued functions from Ω to X which is uniformly μ-a.e. convergent on Ω to x.

1.2. The Bochner Integral

As in the preceding section, let X be a real Banach space, (Ω, Σ, μ) a σ-finite and complete measure space and let $x : \Omega \to X$ be a countably-valued function. Then there exist $\{\Omega_n;\, n \in \mathbb{N}\} \subset \Sigma$ and $\{x_n;\, n \in \mathbb{N}\} \subset X$, satisfying $\Omega_k \cap \Omega_p = \emptyset$ for each $k \neq p$, $\Omega = \cup_{n\geq 0}\Omega_n$, and such that $x(\omega) = x_n$ for each $n \in \mathbb{N}$ and each $\omega \in \Omega_n$. Obviously, the two families $\{\Omega_n;\, n \in \mathbb{N}\}$ and $\{x_n;\, n \in \mathbb{N}\}$ which define a countably-valued function are not unique. For this reason, in all that follows, a pair of sets $(\{\Omega_n;\, n \in \mathbb{N}\}, \{x_n;\, n \in \mathbb{N}\})$ enjoying the above properties is called *a representation of the countably-valued function* x. Inasmuch as Ω has σ-finite measure, each countably-valued function $x : \Omega \to X$ admits at least one representation with the property that, for each $n \in \mathbb{N}$, $\mu(\Omega_n) < +\infty$. Such a representation is called *σ-finite representation.*

Definition 1.2.1. Let $x : \Omega \to X$ be a countably-valued function and let $\mathcal{R} = (\{\Omega_n;\, n \in \mathbb{N}\}, \{x_n;\, n \in \mathbb{N}\})$ be one of its σ-finite representations. We say that $\mathcal{R}$ is *Bochner integrable* (B-integrable) on Ω with respect to μ, if

$$\sum_{n=0}^{\infty} \mu(\Omega_n)\|x_n\| < +\infty.$$

Remark 1.2.1. If $\mathcal{R}$ and $\mathcal{R}'$ are two σ-finite representations of a countably-valued function $x : \Omega \to X$, the series $\sum_{n=0}^{\infty} \mu(\Omega_n)x_n$ and $\sum_{n=0}^{\infty} \mu(\Omega'_n)x'_n$ are either both convergent, or both divergent, in the norm topology of X, and, in the former case, they have the same sum. Accordingly, $\mathcal{R}$ is B-integrable on Ω with respect to μ if and only if $\mathcal{R}'$ enjoys the same property.

This remark enables us to introduce:

Definition 1.2.2. The countably-valued function $x : \Omega \to X$ is *Bochner integrable* on Ω with respect to μ if it has a σ-finite representation

$$\mathcal{R} = (\{\Omega_n;\, n \in \mathbb{N}\}, \{x_n;\, n \in \mathbb{N}\})$$

which is B-integrable on Ω with respect to μ. In this case, the vector

$$\sum_{n=0}^{\infty} \mu(\Omega_n)x_n = \int_\Omega x(\omega)\, d\mu(\omega) = \int_\Omega x\, d\mu,$$

which does not depend on the choice of $\mathcal{R}$ (see Remark 1.2.1), is called *the Bochner integral* on Ω of the function x with respect to μ.

Definition 1.2.3. A function $x : \Omega \to X$ is *Bochner integrable* on Ω with respect to μ if it is strongly measurable and there exists a sequence of countably-valued functions $(x_k)_{k\in\mathbb{N}}$, Bochner integrable on Ω with respect to μ, such that

$$\lim_{k\to\infty} \int_\Omega \|x(\omega) - x_k(\omega)\|\, d\mu(\omega) = 0.$$

Proposition 1.2.1. *If $x : \Omega \to X$ is Bochner integrable on Ω with respect to μ and $(x_k)_{k\in\mathbb{N}}$ is a sequence with the properties in Definition* 1.2.3, *then there exists*

$$\lim_{k\to\infty} \int_\Omega x_k\, d\mu$$

in the norm topology of X. In addition, if $(y_k)_{k\in\mathbb{N}}$ is another sequence of countably-valued functions with the property that

$$\lim_{k\to\infty} \int_\Omega \|x(\omega) - y_k(\omega)\|\, d\mu(\omega) = 0,$$

then

$$\lim_{k\to\infty} \int_\Omega x_k(\omega)\, d\mu(\omega) = \lim_{k\to\infty} \int_\Omega y_k(\omega)\, d\mu(\omega).$$

Proof. Let $\varepsilon > 0$ and let $k(\varepsilon) \in \mathbb{N}$ be such that

$$\int_\Omega \|x - x_k\|\, d\mu \le \varepsilon$$