ESM 9

THE THEORY OF GENERAL ECONOMIC EQUILIBRIUM:
A DIFFERENTIABLE APPROACH

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The theory of general economic equilibrium

A differentiable approach

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Preface

The title of this book describes its content, as does the Introduction with more detail. In a sentence it could be summarized as a report on the theoretical developments that followed the publication in 1970 of the fundamental paper by G. Debreu, "Economies with a Finite Set of Equilibria."

I cannot pretend that this book does not assume anything. The economics is essentially self-contained, but it would not be sensible to attempt a systematic reading without a good grounding in the differentiable calculus of several variables and the basic elements of real analysis. Although they are not the central mathematical techniques of this book, it is nevertheless true that some background in topology, measure theory, and convex analysis can be of considerable help. This said, I wish to emphasize that a constant effort is made to facilitate comprehension. Thus, Chapter 1 is devoted entirely to the mathematics, different chapters and sections do not need to be read consecutively, and digressions are written so that they can be skipped. The same is true of proofs (their beginnings and ends are always clearly marked by, respectively, the term *Proof* and the symbol ■). Examples and figures abound, intuitive arguments are given often, essential results are proved leisurely and sometimes repeatedly. Even at the cost of an occasional pedestrian proof the use of comparatively sophisticated techniques is avoided.

This book was written over a period longer than I wish to remember. It was conceived and started at Berkeley, continued at the Universitat Autònoma de Barcelona, and concluded at Harvard. Each of these institutions provided a wonderfully supportive environment. It is only fair, however, to single out the very special contribution of Berkeley. After all, the book is an outgrowth of the intense research atmosphere on the differentiable approach to economics fostered there in the seventies by G. Debreu and S. Smale. My first attempt at a systematic presentation was a course taught in the Berkeley Mathematics Department in the spring of 1976. The typewritten notes for this course had a limited distribution and constitute the (very distant) antecessor of this book.

My debt to G. Debreu will be obvious enough through the book. Let me just say here that without his scientific work this book could not have been written, and without his friendly and kind encouragement it may not have been.

My ideas on the uses of the calculus in economics have been shaped by the interaction with many colleagues and coauthors. The fundamental roles of G. Debreu and S. Smale I have already mentioned. But I should also explicitly acknowledge A. Araujo, Y. Balasko, H. Cheng, E. Dierker, H. Dierker, W. Hildenbrand, T. Kehoe, W. Neuefeind, H. Scarf, S. Schecter, D. Sondermann, and H. Sonnenschein. In a sense this book follows in the tradition of the lecture notes by E. Dierker, *Topological Methods in Walrasian Economics* (1974). I only wish I could have kept the high standards of elegance and lucidity set by him.

I am grateful to a number of people who have helped me in one form or another to improve different parts of the book. Among them are B. Allen, R. Anderson, G. Chatterjee, H. Cheng, B. Cornet, G. Debreu, E. Dekel, D. Duffie, B. Grodal, F. Hahn, O. Hart, S. Hart, C. Langlois, J. Nachbar, L. T. Nielsen, C. Simon, N. Singh, W. Trockel, and D. Wells. Special thanks are due to J. Nachbar for his help with the references and the index. The figures were prepared by W. Minty. Finally, a word of thanks to the very able people at Cambridge University Press. A thought should also go to my wife, Esther, who has had to endure the process of writing this book to an extent larger than is fair.

Over the years I have enjoyed the financial support of a Sloan Fellowship and of the National Science Foundation. I am grateful for it.

The manuscript for this book has been prepared in the old and hard way, that is to say, by incessant typing and retyping. In this respect I have been fortunate to count at different stages on the able collaboration of I. Workman, K. Kraco, M. Bosco, and G. Cogswell.

Andreu Mas-Colell

Glossary

This is a glossary of symbols and abbreviations. For terms such as *manifold, rank,* and *kernel,* consult the analytical index (where the first page reference will usually provide a definition).

Mathematics

$\{a \in X : *\}$	the subset of X formed by the elements satisfying the property $*$
$\setminus$	set difference; for example, $A \setminus B = \{a \in A : a \notin B\}$
$\times$	product sign
$\#A$	number of elements in the set A
$\prod_{j \in J} A_j$	product of the sets A_j, $j \in J$
A^n	product of n copies of the set A
$f: A \to B$	function with domain A and range contained in B
$a \mapsto f(a)$	a is taken to $f(a)$ by the function f
$f \mid A$	restriction of f to the set A
$f(V)$	image by f of the set V
$f^{-1}(V)$	inverse image under f of the set V
f^{-1}	inverse function of f
$f \circ g$	composition of the functions f and g
∂X, Bdry X	boundary of the set X
cl X	closure of the set X
Int X	interior of the set X
$\mathcal{K}(X)$	space of nonempty compact subsets of X
$\mathcal{C}(X)$	space of nonempty closed subsets of X
$\to$	tends to
lim	limit
Ls, Li	see A.5

u.h.c.	upper hemicontinuous; see A.6
$\chi(X)$	Euler characteristic of A; see A.6
$B_\epsilon(X)$	ϵ neighborhood of the set X
R	set of real numbers
$[a,b]$, $[a,b)$, $(a,b]$, (a,b)	closed, half-closed, open intervals
ab	product of the real numbers a and b
sign a	$+1$, 0, or -1 according to $a >, =, < 0$
a/b, $\frac{a}{b}$	quotient of the real numbers a, b
$\|a\|$	absolute value of the real number a
exp	exponential function
ln	natural logarithm
sup, inf	supremum, infimum
max, min	maximum, minimum
$o(a)$	$o(a)/a \to 0$ as $a \to 0$
$\sum_{j\in J}$, $\sum_{i=1}^{n}$	sum over the indices $j \in J$ or from 1 to n. If J or n is clear in context, it may be dropped.
$\int f(s)\,d\nu(s)$, $\int f\,d\nu$, $\int f$	integral of f with respect to ν
χ_A	indicator function of the set A
$\int_A$	integral over A
supp ν	support of the measure ν; see E.3
$\lambda \circ f^{-1}$	measure induced in the range of f by the measure λ in the domain; see E.2
λ	if used for a measure, it is always Lebesgue measure
dim	dimension
$L^\perp$	perpendicular complement to the subspace L
R^n	n-dimensional Euclidean space
$x = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}$	vector in R^n; sometimes it is written as a row
x^i	ith entry of the vector x

$x \geq y$	$x^i \geq y^i$ for all i
$x > y$	$x \geq y$ and $x \neq y$
$x \gg y$	$x^i > y^i$ for all i
0	origin of R^n
R^n_+	nonnegative orthant of R^n; that is, $\{x \in R^n : x \geq 0\}$
R^n_{++}	strictly positive orthant of R^n; that is, $\{x \in R^n : x \gg 0\}$
x^T	row vector transpose of x
$+, -$	vector addition and subtraction
αx	scalar multiplication of the vector x by the real α
e_i	ith unit vector; that is, $e_i^i = 1$, $e_i^j = 0$ for $j \neq i$
e	$e^i = 1$ for all i
co A	convex hull of the set A
$x \cdot y$	inner product of the vectors x, y
$A(v, v')$	evaluation of the bilinear form A at v, v'
$\|x\|$	norm of the vector x
S^{n-1}	$(n-1)$ unit sphere; that is, $\{x \in R^n : \|x\| = 1\}$
$[a_{ij}]$	matrix with generic entry a_{ij}
A^T	transpose of the matrix A
I	identity matrix
A^{-1}	inverse of the matrix A
$\det A$, $\|A\|$	determinant of A
C^r, $r \geq 0$	r times continuously differentiable
∂f	derivative of the C^1 funtion $f: U \to R^n$, $U \subset R^n$
$\partial_x f$	derivative with respect to x
$\partial f(x)$	evaluation of the derivative at x, a linear map from R^n into R^m. Also, the $m \times n$ matrix of this map (given a base). In the $m = 1$ case the same symbol is used for the $n \times 1$ gradient vector at x
$\partial^2 f$	second derivative of f
$\partial^2 f(x)$	evaluation of the second derivative of f at x, a bilinear form
IFT	implicit function theorem
$T_x M$	tangent space to the manifold M at x
TM	tangent bundle of M; see H.1
$f \pitchfork Z$	f is tranversal to Z; see I.2
$Y \pitchfork Z$	Y is transversal to Z; see I.2
$f \pitchfork_N Z$	f is transversal to Z on N; see I.2
deg	degree
■	end-of-proof sign

Economics

We gather here the symbols that have a single (or at least predominant) meaning throughout all the economic chapters. Hence, this is not a complete list of important concepts. A specific economic notion (e.g., the excess utility map) is always denoted by the same symbol [in this case, $h(\lambda)$], but if the concept is particular to a chapter (in this case, Chapter 5), the symbol may be used for a different purpose in another chapter.

ℓ	number of commodities
R^ℓ	commodity space
p	price vector
S	$\{p \in S^{\ell-1} : p \gg 0\}$
X	consumption set; usually it is one of the sets R^ℓ_+, $R^\ell_+ \setminus \{0\}$, R^ℓ_{++}
$\succsim$	preference relation; see 2.2.1
$\succ$, $\sim$	preference and indifference relation derived from $\succsim$
$b(p, w)$	budget set at price vector p and wealth w; see 2.2.1
ω	initial endowment vector
$\varphi(\cdot)$	demand function; the argument of the function may vary according to the context
$f(\cdot)$	excess demand function; it may be individual or aggregate
$g_{\succsim}(x)$	unit normal to $\{z : z \sim x\}$ at x in the direction of preference; see 2.3
$c(\succsim, x)$	Gaussian curvature of Bdry$\{z : z \succsim x\}$ at x; see 2.5
$S(p, w)$	substitution matrix; see 2.7
$\mathcal{P}_{\text{cl}}$	space of continuous, monotone preferences on $R^\ell_+ \setminus \{0\}$; see 2.4
$\mathcal{P}_b$	space of continuous, monotone preferences on R^ℓ_{++} with $\{z : z \succsim x\}$ closed in R^ℓ for all x; see 2.4
$\mathcal{P}$	generic symbol for either $\mathcal{P}_{\text{cl}}$ or $\mathcal{P}_{\text{b}}$; see 2.4
$\mathcal{P}_c$	$\{\succsim \in \mathcal{P} : \succsim \text{ is convex}\}$; see 2.6
$\mathcal{P}^r$, $r \geq 0$	space of C^r preferences in $\mathcal{P}$; see 2.4
$\mathcal{P}^r_c$, $r \geq 0$	space of C^r, convex preferences in $\mathcal{P}$; see 2.6
$\mathcal{P}^r_{sm}$, $r \geq 0$	space of C^r, strictly monotone (differentiable strictly monotone if $r > 0$) preferences in $\mathcal{P}$; see 2.4

$\mathcal{P}^r_{sc},\ r \geq 0$	space of C^r, strictly convex (differentiable strictly convex if $r > 1$) preferences in $\mathcal{P}$; see 2.6
Y	production set; see 3.3
Y^*	$\{p \in S: \sup p \cdot Y < \infty\}$; see 3.4
γ_Y	distance function; see 3.4
π_Y	projection function; see 3.4
Γ_Y	normal mainfold; see 3.4
β_Y	profit function; see 3.4
P	set of Pareto optimal allocations; see 4.6
I	set of consumers, either a finite set or $[0,1]$; see 5.2
t	generic element of I
$\mathcal{A}$	generic symbol for a space of characteristics; that is, a product of a space of preferences and an initial endowments set; see 5.2
$\mathcal{E}: I \to \mathcal{A}$	an economy: see 5.2
$\mathfrak{M}$	space of economies; see 5.8
$\mathfrak{M}^*$	space of regular economies; see 5.8
$W(\mathcal{E})$	set of equilibrium prices for the economy $\mathcal{E}$; see 5.8
index p	see 5.3.5

Contents

Introduction

1 Modern general equilibrium theory and the calculus

Until the end of World War II mathematical economics was almost synonymous with the application of differential calculus to economics. It was on the strength of this technique that the mathematical approach to economics was initiated by Cournot (1838) and that the theory of general economic equilibrium was created by Walras (1874) and Pareto (1909). Hicks's *Value and Capital* (1939) and Samuelson's *Foundations of Economic Analysis* (1947) represent the culmination of this classical era.

After World War II general equilibrium theory advanced gradually toward the center of economics, but the process was accompanied by a dramatic change of techniques: an almost complete replacement of the calculus by convexity theory and topology. In the fundamental books of the modern tradition, such as Debreu's *Theory of Value* (1959), Arrow and Hahn's *General Competitive Analysis* (1971), Scarf's *Computation of Equilibrium Prices* (1982), and Hildenbrand's *Core and Equilibria of a Large Economy* (1974), derivatives either are entirely absent or play, at most, a peripheral role.

Why did this change occur? Appealing to the combined impact of Leontief's input–output analysis, Dantzig and Koopmans's linear programming, and von Neumann and Morgenstern's theory of games, would be correct but begs the question. Schematizing somewhat (or perhaps a great deal), we could mention two internal weaknesses of the traditional calculus approach that detracted from its rigor and, more importantly, impeded progress.

The first weakness was the custom of settling the problem of existence of an equilibrium by counting equations and unknowns, that is, by verifying that they were the same in number. It became obvious with time, and it never quite escaped the most perceptive authors, that this simply would not do. Eventually, topology came to the rescue and in the form of the fixed-point theory emerged as one of the technical cornerstones of the modern approach.

The second weakness was the inability of the traditional analysis to handle inequalities in a satisfactory manner or even to realize that there was an inequality problem at all. It was here that convexity theory proved to be a decisive tool.

2 The return of the differentiability approach

In 1970 G. Debreu published a pathbreaking paper on economies with a finite set of equilibria. This contribution was at the start of a renewed interest in differentiability techniques. It marked the introduction into economics of the methods of differential topology (particularly of the generic point of view and the theory of transversality of maps) developed by mathematicians, such as R. Thom, during the fifties and early sixties. It would also be difficult to overstress the impact of the delightful little book by Milnor, *Topology from the Differentiable Viewpoint* (1965), on the popularization of the new techniques outside of pure mathematics.

The rebirth of interest in the calculus was born of the perception that the determinateness of equilibrium problem is not exhausted by the existence question but should include matters such as local uniqueness, persistency of equilibria under small perturbations of the data, comparative statics, and stability properties. It had long been recognized (e.g., in the study of stability) that smoothness hypotheses were required to handle some of these issues. The novelty was the realization that the mathematical theory of transversality offered a framework for the rigorous application of the traditional counting-of-the-equations-and-unknowns method to the analysis, not of the existence question, but of other central aspects of the determinateness problem, namely the local uniqueness and the persistency (i.e., the "structural stability") of the equilibria.

Very roughly, the relevance of the transversality theorems to the counting of equations and unknowns could be described as follows. If anything, a count of equations and unknowns must be an appeal to the implicit function theorem (IFT). It is this that allows, locally, the expression of a solution as a differentiable function of exogenous parameters. But the implicit function theorem requires that a "regularity" condition be satisfied: The system of equations should be independent at the solution. To simply assume this independence is not legitimate because solutions are endogenously determined. The transversality theorems offer the way out of the dilemma. They permit us to assert that, in a certain precise sense, the independence condition, while not always satisfied, will typically (or "generically") be so. In ways that would be difficult to summarize in one line, but that are sometimes quite unexpected, the combination of the implicit function theorem and the transversality theorems turns out to be remarkably powerful.

Once the new differential methods had been applied with great success to the solution of basic problems that could not be handled to any degree without them, it is only natural that researchers proceeded to a reexamination of the classical areas of general equilibrium theory from a differentiable perspective. The spell had been broken, so to speak. The gain from this task has been substantial. A general sharpening of the theory has been obtained (e.g., rate of convergence results for limit theorems in economies with many agents) and, occasionally, a fundamentally new insight (e.g., on the topic of computing equilibria).

3 Object and content of the book

The one-line summary of this book is that it gives an account of the theoretical developments just described. Many authors have participated in this work. The pioneering role of G. Debreu has already been mentioned. It is only fair, also, to underline the contributions of S. Smale, E. Dierker, and Y. Balasko.

This is a book of two minds. It is, on the one hand, a book on technique, and technique is, undoubtedly, its raison d'être. Topics and examples have often been chosen for no other reason than their illustrative value for some aspect of the differentiability techniques.

But the book also uses technique as an excuse for a systematic, although not exhaustive, account of some of the central topics of general equilibrium theory. In other words, it is one of the objectives of this book to reexamine the theory of general economic equilibrium from the differentiable point of view.

We give now a brief chapter-by-chapter summary of the content of the book. The "Introduction" section in each of Chapters 2 through 8 contains a more detailed description.

Chapter 1 gathers the mathematics used throughout the book. It covers not only differentiability but also topics such as measure theory, Lipschitzian functions, and convexifying effects of averaging sets. Chapter 1, to put it briefly, is in the nature of a technical appendix.

Chapters 2 and *3* contain the smooth description of our economic agents: consumers and firms, respectively.

Chapter 4 deals with the theory of Pareto optimality. Obviously, issues traditionally associated with the calculus approach – for example, first-