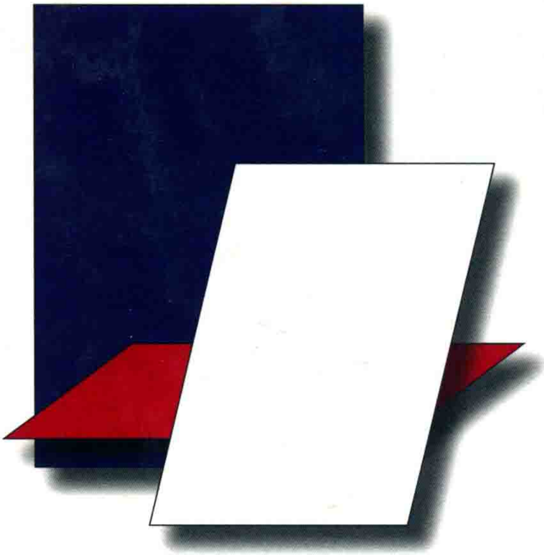

Nonnegative Matrices, Positive Operators, and Applications

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Dedicated to our respective families

Preface

The theory and methods of nonnegative matrices and positive operators have found more applications in science, engineering, and technology during the past decades. Traditionally, nonnegative matrices have direct applications in Markov chains, game theory, iterative methods for solving linear algebraic equations, and etc. A recent example of nonnegative matrices is the so-called Google matrix, the world's largest one, for the computation of the PageRank used by the Google Web search engine [Bryan and Leise (2006); Langville and Meyer (2006)]. The Google matrix is a stochastic matrix, that is, it is a nonnegative square matrix each row of which sums to be one. Because of its huge size presently in the order of at least 10 billions, seeking faster and more efficient numerical algorithms for the PageRank computation of the Google matrix is a current research topic.

On the other hand, mathematical modeling of complicated physical and social systems often leads to the statistical study of deterministic dynamical systems, which may be the main practical and reliable approach to obtaining a meaningful and reasonable solution to the problem [Albert and Barabasi (2002); Dorogovstev and Mendes (2002); Newman (2003); Ding and Zhou (2009)]. For example, for a given chaotic discrete dynamical system $S : X \to X$ on a state space X, instead of observing the deterministic properties of the individual sequences of the iterates which exhibit sensitive dependence on initial conditions, one can consider the probabilistic distribution of such eventually unpredictable or random-like iterates. Functional analysis plays a key role for the mathematical investigation of the dynamics from the global point of view. In particular, infinite dimensional positive operators are introduced to describe the evolution of the probability density functions associated with the underlying dynamical system.

An extremely important class of positive operators is that of Markov operators which transform probability density functions into themselves. Markov operators appear in the stochastic analysis of deterministic dynamical systems. They are widely used in Markov processes, stochastic differential equations, and applied probability. A subclass of Markov operators consists of Frobenius-Perron operators that describe the evolution of statistical quantities associated with measurable transformations. There are two important and related problems concerning Frobenius-Perron operators. One is about the existence of stationary density functions of the operator, which give absolutely continuous invariant probability measures that describe the statistical properties of the dynamics, such as the probability distribution of the sequences of the iterates for almost all initial points and the rate of the decay of correlations of such sequences. The other problem, which is more important in applications, is about efficient computation of such invariant probability measures to any prescribed precision. The basic idea behind the numerical analysis of Frobenius-Perron operators is to approximate the infinite dimensional Markov operator with finite dimensional linear operators that keep the structure of the approximated operator. Here, positivity is the structure of the original operator. Thus, the resulting finite dimensional Markov operator can be represented by a nonnegative (usually stochastic) matrix with respect to any basis consisting of density functions, so the theory and methods of nonnegative matrices and the theory of positive operators can work together to address the computational challenges of such important positive operators.

This textbook aims at providing the fundamental theory and methods of nonnegative matrices and positive operators, and their modern applications in several applied fields, such as the life science, electrical engineering, and the Web information retrieval technology. Naturally the book is divided into two parts. Part I consists of Chapters 1 to 4 concerning nonnegative matrices and their applications, and Chapters 5 through 11 constitute Part II on positive operators, their finite dimensional approximations via nonnegative matrices, and several representative applications including modern ones. There are excellent textbooks and monographs on nonnegative matrices and positive operators, respectively, such as [Berman and Plemmons (1979); Minc (1988); Bapat and Raghavan (1997)] for the former and [Schaefer (1974); Walters (1982); Aliprantis and Burkinshaw (1985); Krengel (1985); Lasota and Mackey (1994); Boyarsky and Góra (1997); Zaanen (1997); Baladi (2000)] for the latter. But due to ever growing applications of nonnegative matrices in physical sciences, engineering, technology, and social sciences, and in the numerical analysis of positive operators arising from various areas of computational science, we feel a need for writing the present textbook that combines the two topics, which are closely related but often separately presented, into a single unity. Because of the intended large readership, we have kept in mind the consideration of the material coverage and tried to balance the presentation of abstract theories and practical methods to benefit the readers studying or working in different disciplines. For example, we only give an introduction to the theory of abstract positive operators to lessen an unnecessary burden of the reader; on the other hand, we present in more detail the theoretical and numerical analysis of several concrete classes of positive operators, such as Markov operators, Frobenius-Perron operators, and composition operators widely used in applied fields, so that most of the book can be understood by as many readers as possible with various backgrounds. Thus, lots of more general and advanced theorems on abstract positive operators are not included, but they can be found in monographs on Banach lattices and positive operators.

Specifically, the textbook has eleven chapters, each of which has five sections. Chapter 1 provides basic algebraic, spectral, and combinatorial properties of nonnegative matrices, and we also study an important class of matrices whose inverse is a nonnegative matrix. Chapter 2 is the main body of the theory of nonnegative matrices, in which we shall study the classic Perron-Frobenius theory for irreducible nonnegative matrices. In Chapter 3 we focus on stochastic matrices, a special class of nonnegative matrices with spectral radius 1 which are closely related to any field involving probability. Applications of nonnegative matrices to Markov chains, the Web information retrieval, dynamical geometry, and the iterative method for solving linear equations are discussed in Chapter 4. Chapter 5 begins the study of positive operators, and we are devoted to the general properties of abstract positive operators on ordered vector spaces and in particular Banach lattices. The spectral analysis and ergodic theory of positive operators are the main themes of Chapter 6, which are of fundamental importance in applications of positive operators in physical sciences. Markov operators, which play a main role in stochastic processes and the statistical analysis of deterministic dynamical systems, will be introduced in Chapter 7 and the asymptotic properties of their iterates will be investigated for a class of Markov operators, based on a powerful decomposition theorem. A special class of Markov operators, called Frobenius-Perron operators which are used for the existence and computation of absolutely continuous invariant probability measures of nonsingular transformations of measure spaces, will be explored in detail in Chapter 8. We shall present some existence theorems for several classes of one or multi-dimensional transformations. Topological dynamical systems will be briefly studied in Chapter 9 since composition operators defined on the Banach lattice of continuous functions on a compact metric space are used for exploring the asymptotic behavior of continuous transformations. In Chapter 10 we study structure-preserving approximations of Markov operators and in particular Frobenius-Perron operators via finite dimensional Markov operators. The convergence of Ulam's piecewise constant method and the piecewise linear Markov method will be presented for several classes of transformations, and their error estimates will be given for a class of interval mappings. The last chapter provides several representative applications of positive operators to approximating continuous functions, evolutionary partial differential equations, computational molecular dynamics, and wireless communications.

This textbook can be used for students in mathematical sciences as an upper level undergraduate or beginning graduate course on nonnegative matrices or/and positive operators. Besides, the first four chapters can be used for a one semester course on the theory of nonnegative matrices, and some other ones can form a course on positive operators or applied ergodic theory. Materials of the book can also be adopted as a textbook for a specialized course in different areas of physical sciences with a purpose of mathematics enhancement for the interdisciplinary research. For students or researchers in engineering subjects such as electrical engineering, who are interested in applied ergodic theory, this textbook can be used for a graduate special topic course or a mathematical tool book. A good background of linear algebra and functional analysis is sufficient for understanding an essential part of the book, but we have included Appendix A for basic matrix theory and Appendix B for fundamental functional analysis for the reader's convenience. Some of the exercises in each chapter are complements to the main text, so the reader should try as many as possible, or at least take a look and read appropriate references if possible. In writing this textbook and selecting exercise problems, we have consulted many books and research papers on nonnegative matrices, positive operators, and more specialized topics, including those mentioned above as well as ours. We thank these authors and the others for their works' influences on ours.

Some parts of the textbook, such as approximations of Frobenius-Perron operators in Chapter 10, contain our joint research results in the past sixteen years. Jiu Ding owes a deep gratitude to his Ph.D. thesis advisor, Professor Tien-Yien Li, for introducing him into the field of computational ergodic theory. Aihui Zhou is very grateful to his Ph.D. thesis advisor, Professor Qun Lin, for his constant encouragement in the past two decades. Our research collaboration and writing of this book have been supported by the National Natural Science Foundation of China, the National Basic Research Program of China, the Academy of Mathematics and Systems Science and the State Key Laboratory of Scientific and Engineering Computing at the Chinese Academy of Sciences, the Chinese Ministry of Education, K. C. Wong Education Foundation of Hong Kong, and the A. K. Lucas Endowment at the University of Southern Mississippi, for which we express our sincere thanks.

Just at the completion of this work, we are deeply saddened to learn the untimely death of Dr. Xiangzhong (Jerry) Yang at 49 on February 5, whose *China Bridge Foundation* at the University of Connecticut has supported our joint research. This great scientist will be in our memory forever.

Some chapters from the initial draft of the textbook were used for a graduate course, Topics in Algebra, at the University of Southern Mississippi for the spring semester of 2007. Jiu Ding thanks the students Rebecca Eckhoff, Johnathan McEwen, Ryanne McNeese, Margaret Moore, Kedar Nepal, Corwin Stanford, Anita Waltman, Erin Westmoreland, and Nina Ye in the class for making it fun to teach with success. Several topics and materials in the textbook have been addressed in a graduate course, Numerical Analysis, at the Graduate University of the Chinese Academy of Sciences since 2005. Aihui Zhou would like to acknowledge the many students who took the course for giving him an opportunity to test our ideas.

Finally, we thank the editors Lai Fun Kwong and Yubing Zhai of the World Scientific for their editorial assistance and help, which has made writing and publishing this textbook a pleasant experience.

Jiu Ding and Aihui Zhou, February 11, 2009

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Contents

Chapter 1

Elementary Properties of Nonnegative Matrices

The theory of nonnegative matrices is a specialized topic in matrix theory. Nonnegative matrices can be viewed as finite dimensional positivity-preserving linear operators defined on Euclidean spaces and can be used for structure-preserving approximations of infinite dimensional positive operators in the numerical analysis of such important linear operators with many applications. The first four chapters of this textbook will provide a prototype of general positive operators to be studied later. We assume that the reader has taken a first course in linear algebra and is familiar with the fundamental theory of matrices, which and some more advanced concepts are included in Appendix A for reference.

In the first chapter, we present elementary properties and some basic results for nonnegative matrices. In Section 1.1 we give various algebraic properties of nonnegative matrices, and Section 1.2 is devoted to their spectral properties. As a prerequisite for the next chapter, combinatorial properties of nonnegative matrices and several related concepts will be introduced in Section 1.3. An important class of square matrices whose inverses are nonnegative ones will be studied in Section 1.4 because of their wide applications in various areas.

1.1 Algebraic Properties of Nonnegative Matrices

Because of the *total ordering* property of the set $\mathbb{R}$ of all real numbers, that is, exactly one of the relations $x < y$, $y < x$, or $x = y$ holds for any two real numbers x and y, it is convenient to be able to compare n-dimensional (column) vectors in the n-dimensional Euclidean space $\mathbb{R}^n$ component by component and $m \times n$ matrices in the mn-dimensional vector space $\mathbb{R}^{m\times n}$ entry by entry, thus providing a *canonical ordering* relation for real vectors and real matrices. A component-wise *partial ordering* on $\mathbb{R}^n$ is defined by the following relation

$$x \leq y \quad \text{if and only if} \quad x_i \leq y_i, \quad i = 1, \cdots, n. \tag{1.1}$$

Two vectors $x, y \in \mathbb{R}^n$ are said to be *comparable* under the ordering if $x \leq y$ or $y \leq x$ which can also be written as $x \geq y$. The following properties of the natural ordering can be immediately verified, the first three of which explain why the relation $\leq$ is a partial ordering [Zaanen (1997)].

Proposition 1.1. *The canonical ordering relation $\leq$ defined on $\mathbb{R}^n$ by (1.1) satisfies the properties:*

(i) $x \leq x$ *for all* $x \in \mathbb{R}^n$ (reflexivity);
(ii) $x \leq y$ *and* $y \leq x$ *imply* $x = y$ (transitivity);
(iii) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ (anti-symmetry);
(iv) $x \leq y$ *implies* $ax \leq ay$ *for any number* $a \geq 0$ *and* $ax \geq ay$ *if* $a \leq 0$;
(v) *if* $x \leq y$, *then* $x + z \leq y + z$ *for all vectors* $z \in \mathbb{R}^n$.

If a vector $x \in \mathbb{R}^n$ is such that $x \geq 0$, then x is said to be *nonnegative*, and the set of all nonnegative vectors in $\mathbb{R}^n$ is a cone, called the *positive cone* of $\mathbb{R}^n$ and is denoted by $\mathbb{R}^n_+$. Thus, if $x, y \in \mathbb{R}^n_+$, then $x + y \in \mathbb{R}^n_+$ and $ax \in \mathbb{R}^n_+$ for all nonnegative numbers a. The special vector e of all components 1 is the *order unit* of $\mathbb{R}^n$ which is the *ideal* generated by e (see Section 5.3 for general definitions of such terms). The *absolute value vector* of any complex (column) vector x in the n-dimensional unitary space $\mathbb{C}^n$ is defined to be the nonnegative vector

$$|x| = (|x_1|, \cdots, |x_n|)^T$$

in $\mathbb{R}^n_+$. It is again straightforward to verify the following properties of absolute value vectors of complex ones.

Proposition 1.2. *Let* $x, y \in \mathbb{C}^n$. *Then*

(i) $|x| \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$;
(ii) $|ax| = |a||x|$ *for all complex numbers* a;
(iii) $|x + y| \leq |x| + |y|$ (triangle inequality).

Note that properties (i)-(iii) in Proposition 1.2 for the absolute value operation $|\ |$ look similar to those for a norm $\|\ \|$ in format, but the difference is that while absolute value vectors are nonnegative vectors, norms are nonnegative numbers.

A norm $\|\ \|$ on $\mathbb{R}^n$ is called a *lattice norm* if for any $x, y \in \mathbb{R}^n$,

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\|. \tag{1.2}$$

It can be shown that condition (1.2) is equivalent to the equality

$$\| \, |x| \, \| = \|x\|, \ \ \forall \, x \in \mathbb{R}^n.$$

Thus the p-norm $\| \, \|_p$ on $\mathbb{R}^n$ with $1 \le p \le \infty$ is a lattice norm.

A matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ of all $m \times n$ real matrices is said to be *nonnegative* if $a_{ij} \geq 0$ for all indices i and j, and this is denoted by $A \geq 0$. Similarly A is called a *positive matrix* and is written as $A > 0$ if all $a_{ij} > 0$. The set of all nonnegative matrices in $\mathbb{R}^{m \times n}$ is denoted by $\mathbb{R}_+^{m \times n}$. Given any two real matrices A and B of the same order, we write $A \geq B$ (or equivalently, $B \leq A$) if $A - B \geq 0$, and $A > B$ if $A - B > 0$. Thus, $A \geq B$ (or $A > B$) means that $a_{ij} \geq b_{ij}$ (or $a_{ij} > b_{ij}$) for all pairs (i, j) of row and column indices. It is easy to see that If $A \leq B$ and $C \geq 0$, then

$$AC \leq BC \quad \text{and} \quad CA \leq CB$$

whenever the matrix multiplication is well-defined. It follows that when A and B are nonnegative square matrices, the inequality $A \leq B$ implies that $A^k \leq B^k$ for all nonnegative integers k.

Let $A \in \mathbb{C}^{m \times n}$, the space of all $m \times n$ complex matrices. Then the corresponding *absolute value matrix* is defined to be

$$|A| = (|a_{ij}|) \in \mathbb{R}_+^{m \times n}.$$

Propositions 1.1 and 1.2 also hold for matrices. Moreover, we have

$$|Ax| \leq |A| \, |x|, \ \ \forall \, A \in \mathbb{C}^{m \times n}, \ x \in \mathbb{C}^n$$

and

$$|AB| \leq |A| \, |B|, \ \ \forall \, A \in \mathbb{C}^{m \times k}, \ B \in \mathbb{C}^{k \times n}.$$

In particular, for any $A \in \mathbb{C}^{n \times n}$ and any nonnegative integer k,

$$|A^k| \leq |A|^k.$$

The partial ordering relation (1.1) on $\mathbb{R}^n$ induces the concept of *monotone operators* on $\mathbb{R}^n$ in the following definition.

Definition 1.1. A mapping $S : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be *isotone* if $S(x) \leq S(y)$ whenever $x \leq y$, $x, y \in D$. Similarly, S is said to be *antitone* if $S(x) \geq S(y)$ for all x, $y \in D$ such that $x \leq y$.

Clearly S is antitone if and only if $-S$ is isotone. The concepts of nonnegative matrices and monotone mappings are closely related as the next proposition shows, which can be proved immediately.

Proposition 1.3. *A matrix $A \in \mathbb{R}^{m \times n}$ is nonnegative if and only if the resulting linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ is isotone, which is true if and only if $Ax \geq 0$ whenever $x \in \mathbb{R}_+^n$.*