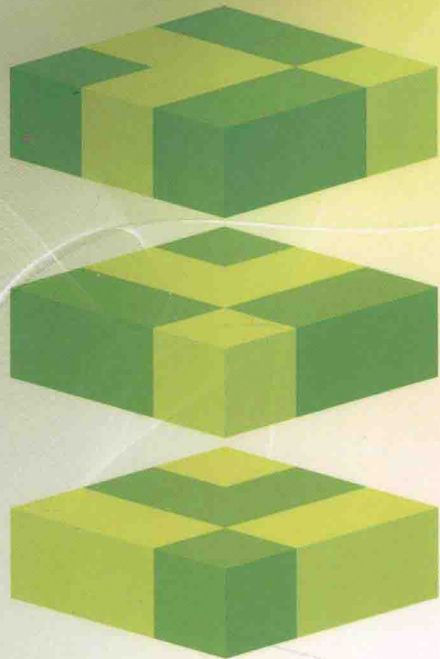


H. PAUL WILLIAMS

Model Building in Mathematical Programming

FIFTH EDITION



 WILEY

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Fifth Edition

H. Paul Williams

London School of Economics, UK



 **WILEY**

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Model Building in Mathematical Programming

To Eileen, Anna, Alexander and Eleanor

Preface

Mathematical programmes are among the most widely used models in operational research and management science. In many cases their application has been so successful that their use has passed out of operational research departments to become an accepted routine planning tool. It is therefore rather surprising that comparatively little attention has been paid in the literature to the problems of formulating and building mathematical programming models or even deciding when such a model is applicable. Most published work has tended to be of two kinds. Firstly, case studies of particular applications have been described in the operational research journals and journals relating to specific industries. Secondly, research work on new algorithms for special classes of problems has provided much material for the more theoretical journals. This book attempts to fill the gap by, in Part I, discussing the general principles of model building in mathematical programming. In Part II, 29 practical problems are presented to which mathematical programming can be applied. By simplifying the problems, much of the tedious institutional detail of case studies is avoided. It is hoped, however, that the essence of the problems is preserved and easily understood. Finally, in Parts III and IV, suggested formulations and solutions to the problems are given together with some computational experience.

Many books already exist on mathematical programming or, in particular, linear programming. Most such books adopt the conventional approach of paying a great deal of attention to algorithms. Since the algorithmic side has been so well and fully covered by other texts, it is given much less attention in this book. The concentration here is more on the building and interpreting of models rather than on the solution process. Nevertheless, it is hoped that this book may spur the reader to delve more deeply into the often challenging algorithmic side of the subject as well. It is, however, the author's contention that the practical problems and model building aspect should come first. This may then provide a motivation for finding out how to solve such models. Although desirable, knowledge of algorithms is no longer necessary if practical use is to be made of mathematical programming. The solution of practical models is now largely automated by the use of commercial package programs that are discussed in Chapter 2.

For the reader with some prior knowledge of mathematical programming, parts of this book may seem trivial and can be skipped or read quickly. Other parts are, however, rather more advanced and present fairly new material. This is particularly true of the chapters on integer programming. Indeed, this book can

be treated in a nonsequential manner. There is much cross-referencing to enable the reader to pass from one relevant section to another. This book is aimed at three types of readers:

1. It is intended to provide students in universities and polytechnics with a solid foundation in the principles of model building as well as the more mathematical, algorithmic side of the subject, which is conventionally taught. For students who finally go on to use mathematical programming to solve real problems, the model building aspect is probably the more important. The problems in Part II provide practical exercises in problem formulation. By formulating models and solving them with the aid of a computer, students learn the art of formulation in the most satisfying way possible. They can compare their numerical solutions with those of other students obtained from differently build models. In this way they learn how to validate a model.

It is also hoped that these problems will be of use to research students seeking new algorithms for solving mathematical programming problems. Very often they have to rely on trivial or randomly generated models to test their computational procedures. Such models are far from typical of those found in the real world. Moreover, they are one (or more) steps removed from practical situations. They therefore obscure the need for efficient formulations as well as algorithms.

2. This book is also intended to provide managers with a fairly nontechnical appreciation of the scope and limitations of mathematical programming. In addition, by looking at the practical problems described in Part II they may recognize a situation in their own organization to which they had not realized mathematical programming could be applied.
3. Finally, constructing a mathematical model of an organization provides one of the best methods of understanding that organization. It is hoped that the general reader will be able to use the principles described in this book to build mathematical models and thereby learn about the functioning of systems, which purely verbal descriptions fail to explain. It has been the author's experience that the process of building a model of an organization can often be more beneficial even than the obtaining of a solution. A greater understanding of the complex interconnections between different facets of an organization is forced upon anybody who realistically attempts to model that organization.

Part I of this book describes the principles of building mathematical programming models and how they may arise in practice. In particular, linear programming, integer programming and separable programming models are described. A discussion of the practical aspects of solving such models and a very full discussion of the interpretation of their solutions is included.

Part II presents each of the 29 practical problems in sufficient detail to enable the reader to build a mathematical programming model using the numerical data. In some cases the origin of the problem is mentioned.

Part III discusses each problem in detail and presents a possible formulation as a mathematical programming model.

Part IV gives the optimal solutions obtained from the formulations presented in Part III. Some computational experience is also given in order to give the reader some feel of the computational difficulty of solving the particular type of model.

It is hoped that readers will attempt to formulate and possibly solve the problems for themselves before proceeding to Parts III and IV.

By presenting 29 problems from widely different contexts the power of the technique of mathematical programming in giving a method of tackling them all should be apparent. Some problems are intentionally “unusual” in the hope that they may suggest the application of mathematical programming in rather novel areas.

Many references are given at the end of the book. The list is not intended to provide a complete bibliography of the vast number of case studies published. Many excellent case studies have been ignored. The list should, however, provide a representative sample which can be used as a starting point for a deeper search into the literature.

Many people have both knowingly and unknowingly helped in the preparation of successive editions of this book with their suggestions and opinions. In particular I would like to thank Gautam Appa, Sheena and Robert Ashford, Martin Beale, Tony Brearley, Ian Buchanan, Colin Clayman, Lewis Corner, Martyn Jeffreys, Bob Jeroslow, Clifford Jones, Bernard Kemp, Ailsa Land, Adolfo Fonseca Manjarres, Kenneth McKinnon, Gautam Mitra, Heiner Müller-Merbach, Bjorn Nygreen, Pat Rivett, Richard Thomas, Steven Vajda and Will Watkins. I must express a great debt of gratitude to Robin Day of Edinburgh University, whose deep computing knowledge and programming ability helped me immensely in building the models for early editions of this book and implementing our design of the modelling system MAGIC. This has now been replaced by the system NEWMAGIC, which has been written by George Skondras. It is available with the optimizer EMSOL. All the models in the book have been built and solved in this system. The models, formulated in Part III, have been modelled in the NEWMAGIC language and are available on the website: www.wiley.com/go/model_building_mathematical_programming

Since the first edition was written, computational power has increased immensely. Solution times for the models are therefore, in most cases, of little relevance and are ignored. A few of the models are still difficult to solve. In these cases, some computational experience is given.

The fourth edition has been enhanced by new sections or models on Constraint Logic Programming, Data Envelopment Analysis, Hydro Electricity Generation,

Milk Distribution and Yield Management in an airline together with state-of-the-art material and extra references on a number of topics and applications. I am very grateful to Kenneth McKinnon for advice and help with these models.

It is gratifying to know how well this book has been received together with the demand for a fourth edition.

Preface to the Fifth Edition

The fifth edition includes new sections on Stochastic Programming and Column Generation, a subsection on the Vehicle Routing problem and an enhanced section on Constraint Logic Programming. In addition, the subsection on modelling non-linear functions and constraints, by integer variables, has been improved with a more versatile formulation. Many other small clarifications and improvements have also been included.

Five new problems have been added: a Car Rental and Return problem and an extension, an Airport Lost Baggage Distribution problem and two problems in Molecular Biology.

New references have been added, although it is recognized that it is impossible to mention all the excellent papers that have been written since earlier editions of this book. I apologize to the many authors of these papers who have not been cited.

A number of people have given me advice on improving and correcting the fourth edition to produce this fifth edition. In particular, I would like to mention Harvey Greenberg, John Hooker, Cormac Lucas, Andrew McGee and Ken McKinnon. They have all helped validate the new material. Also, a number of correspondents have pointed out small typos. I am grateful to them all.

PAUL WILLIAMS
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Contents

Preface

xvii

Part I	1
1 Introduction	3
1.1 The concept of a model	3
1.2 Mathematical programming models	5
2 Solving mathematical programming models	11
2.1 Algorithms and packages	11
2.1.1 Reduction	12
2.1.2 Starting solutions	12
2.1.3 Simple bounding constraints	12
2.1.4 Ranged constraints	13
2.1.5 Generalized upper bounding constraints	13
2.1.6 Sensitivity analysis	13
2.2 Practical considerations	13
2.3 Decision support and expert systems	16
2.4 Constraint programming (CP)	17
3 Building linear programming models	21
3.1 The importance of linearity	21
3.2 Defining objectives	23
3.2.1 Single objectives	24
3.2.2 Multiple and conflicting objectives	26
3.2.3 Minimax objectives	27
3.2.4 Ratio objectives	28
3.2.5 Non-existent and non-optimizable objectives	29
3.3 Defining constraints	29
3.3.1 Productive capacity constraints	29
3.3.2 Raw material availabilities	30
3.3.3 Marketing demands and limitations	30
3.3.4 Material balance (continuity) constraints	30

3.3.5	Quality stipulations	31
3.3.6	Hard and soft constraints	31
3.3.7	Chance constraints	32
3.3.8	Conflicting constraints	32
3.3.9	Redundant constraints	34
3.3.10	Simple and generalized upper bounds	35
3.3.11	Unusual constraints	35
3.4	How to build a good model	36
3.4.1	Ease of understanding the model	36
3.4.2	Ease of detecting errors in the model	37
3.4.3	Ease of computing the solution	37
3.4.4	Modal formulation	38
3.4.5	Units of measurement	40
3.5	The use of modelling languages	40
3.5.1	A more natural input format	41
3.5.2	Debugging is made easier	41
3.5.3	Modification is made easier	41
3.5.4	Repetition is automated	41
3.5.5	Special purpose generators using a high level language	41
3.5.6	Matrix block building systems	42
3.5.7	Data structuring systems	42
3.5.8	Mathematical languages	42
3.5.8.1	SETS	43
3.5.8.2	DATA	43
3.5.8.3	VARIABLES	43
3.5.8.4	OBJECTIVE	43
3.5.8.5	CONSTRAINTS	43
4	Structured linear programming models	45
4.1	Multiple plant, product and period models	45
4.2	Stochastic programmes	53
4.3	Decomposing a large model	55
4.3.1	The submodels	63
4.3.2	The restricted master model	64
5	Applications and special types of mathematical programming model	67
5.1	Typical applications	67
5.1.1	The petroleum industry	68
5.1.2	The chemical industry	68
5.1.3	Manufacturing industry	68
5.1.4	Transport and distribution	69
5.1.5	Finance	69
5.1.6	Agriculture	70

5.1.7	Health	70
5.1.8	Mining	70
5.1.9	Manpower planning	71
5.1.10	Food	71
5.1.11	Energy	71
5.1.12	Pulp and paper	72
5.1.13	Advertising	72
5.1.14	Defence	72
5.1.15	The supply chain	72
5.1.16	Other applications	73
5.2	Economic models	74
5.2.1	The static model	74
5.2.2	The dynamic model	80
5.2.3	Aggregation	81
5.3	Network models	81
5.3.1	The transportation problem	82
5.3.2	The assignment problem	87
5.3.3	The transshipment problem	88
5.3.4	The minimum cost flow problem	89
5.3.5	The shortest path problem	93
5.3.6	Maximum flow through a network	93
5.3.7	Critical path analysis	94
5.4	Converting linear programs to networks	98

6	Interpreting and using the solution of a linear programming model	103
6.1	Validating a model	103
6.1.1	Infeasible models	103
6.1.2	Unbounded models	104
6.1.3	Solvable models	105
6.2	Economic interpretations	107
6.2.1	The dual model	109
6.2.2	Shadow prices	112
6.2.3	Productive capacity constraints	114
6.2.4	Raw material availabilities	114
6.2.5	Marketing demands and limitations	114
6.2.6	Material balance (continuity) constraints	114
6.2.7	Quality stipulations	114
6.2.8	Reduced costs	116
6.3	Sensitivity analysis and the stability of a model	121
6.3.1	Right-hand side ranges	121
6.3.2	Objective ranges	125
6.3.3	Ranges on interior coefficients	128
6.3.4	Marginal rates of substitution	131
6.3.5	Building stable models	132

6.4	Further investigations using a model	133
6.5	Presentation of the solutions	135
7	Non-linear models	137
7.1	Typical applications	137
7.2	Local and global optima	140
7.3	Separable programming	147
7.4	Converting a problem to a separable model	153
8	Integer programming	155
8.1	Introduction	155
8.2	The applicability of integer programming	156
8.2.1	Problems with discrete inputs and outputs	156
8.2.2	Problems with logical conditions	158
8.2.3	Combinatorial problems	158
8.2.4	Non-linear problems	160
8.2.5	Network problems	161
8.3	Solving integer programming models	162
8.3.1	Cutting planes methods	162
8.3.2	Enumerative methods	163
8.3.3	Pseudo-Boolean methods	163
8.3.4	Branch and bound methods	164
9	Building integer programming models I	165
9.1	The uses of discrete variables	165
9.1.1	Indivisible (discrete) quantities	165
9.1.2	Decision variables	165
9.1.3	Indicator variables	166
9.2	Logical conditions and 0–1 variables	172
9.3	Special ordered sets of variables	177
9.4	Extra conditions applied to linear programming models	182
9.4.1	Disjunctive constraints	183
9.4.2	Non-convex regions	184
9.4.3	Limiting the number of variables in a solution	186
9.4.4	Sequentially dependent decisions	186
9.4.5	Economies of scale	187
9.4.6	Discrete capacity extensions	188
9.4.7	Maximax objectives	188
9.5	Special kinds of integer programming model	189
9.5.1	Set covering problems	189
9.5.2	Set packing problems	191
9.5.3	Set partitioning problems	193
9.5.4	The knapsack problem	195
9.5.5	The travelling salesman problem	195
9.5.6	The vehicle routing problem	198

9.5.7	The quadratic assignment problem	199
9.6	Column generation	201
10	Building integer programming models II	207
10.1	Good and bad formulations	207
10.1.1	The number of variables in an IP model	207
10.1.2	The number of constraints in an IP model	211
10.2	Simplifying an integer programming model	218
10.2.1	Tightening bounds	218
10.2.2	Simplifying a single integer constraint to another single integer constraint	220
10.2.3	Simplifying a single integer constraint to a collection of integer constraints	222
10.2.4	Simplifying collections of constraints	226
10.2.5	Discontinuous variables	228
10.2.6	An alternative formulation for disjunctive constraints	229
10.2.7	Symmetry	230
10.3	Economic information obtainable by integer programming	231
10.4	Sensitivity analysis and the stability of a model	238
10.4.1	Sensitivity analysis and integer programming	238
10.4.2	Building a stable model	239
10.5	When and how to use integer programming	240
11	The implementation of a mathematical programming system of planning	243
11.1	Acceptance and implementation	243
11.2	The unification of organizational functions	245
11.3	Centralization versus decentralization	247
11.4	The collection of data and the maintenance of a model	249
Part II		251
12	The problems	253
12.1	Food manufacture 1	253
12.2	Food manufacture 2	255
12.3	Factory planning 1	255
12.4	Factory planning 2	256
12.5	Manpower planning	256
12.5.1	Recruitment	257
12.5.2	Retraining	257
12.5.3	Redundancy	258
12.5.4	Overmanning	258
12.5.5	Short-time working	258