# The Uncertainties of Ideal Theory on Hemirings

（半环的不确定性理想理论）

Jianming Zhan（詹建明）



SCIENCE PRESS
Beijing

# The Uncertainties of Ideal Theory on Hemirings

(半环的不确定性理想理论)

Jianming Zhan (詹建明)



SCIENCE PRESS
Beijing, China

Responsible Editors: Yanchao Zhao and Jingke Li


Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing



ISBN 978-7-03-044309-0

# Preface

Semirings which are regarded as a generalization of rings have been found useful in solving problems in different disciplines of applied mathematics and information sciences because a semiring provides an algebraic framework for modelling. By a hemiring, we mean a special semiring with a zero and with a commutative addition. In applications, hemirings are useful in automata and formal languages. Many aspects of the theory of matrices and determinants over semirings have been studied by Beasley et al. in [36], [37], [315], and others. We note that the ideals of semirings play a central role in the structure theory, however, they do not in general coincide with the usual ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. In this book, we attempt to obtain some analogues ring theorems for semirings. Indeed, many results in rings apparently do not have analogous results in semirings if we only consider their ideals.

In the literature, Henriksen[147] first defined a kind of more restricted ideals in a semiring $S$, namely, the $k$-ideals of $S$ having the property that if $S$ is a ring then a complex in $S$ is a $k$-ideal if and only if it is a ring ideal. Another more restricted class of ideals were given in hemirings by Iizuka[155]. However, in an additively commutative semiring $S$, the ideals of a semiring coincide with the "ideals" of a ring provided that the semiring is a hemiring. We now call this ideal an $h$-ideal of a hemiring $S$. The properties of $h$-ideals and also $k$-ideals of a hemiring were thoroughly investigated by Torre in [208] and by using $h$-ideals and $k$-ideals, Torre established some analogous ring theorems for hemirings.

We know that most of practical problems within the fields of economics, engineering, medical sciences, environmental sciences involve data that containing uncertainties. We can not successfully use traditional mathematical tools because of all kinds of uncertainties existing in these problems. In order to solve these problems, many scientists have put forth some special tools such as probability theory, fuzzy set theory[354], rough set theory[272, 273] and soft set theory[255].

After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. It is also interesting to observe that fuzzy calculus has been adopted in artificial intelligence which involves essentially the "(min, max) semirings" (see [87]). Moreover, we note that the same hemirings can be used to study the fundamental concepts of automata theory such as nondeterminism. Many other applications can be found in the literature on semirings and their applications (see [123]). Recently, Jun[176] considered the fuzzy setting of $h$-

ideals of hemirings. Moreover, Zhan et al.[363] discussed the $h$-hemiregular hemirings by using the fuzzy $h$-ideals and they discussed the properties $L$-fuzzy $h$-ideals with operators in hemirings[363]. As a continuation of this investigation, Yin et al.[347] introduced the concepts of fuzzy $h$-bi-ideals and fuzzy $h$-quasi-ideals of hemirings. By using these fuzzy ideals, some characterization theorems of $h$-hemiregular and $h$-intra-hemiregular hemirings are obtained. Other important results related with fuzzy $h$-ideals of a hemiring were given in [93, 94, 152, 358].

Rough set theory was first proposed by Pawlak[274–276] as an important tool to discuss imprecision, vagueness and uncertainty. The theory shows important applications in intelligent decision making systems, machines learning, cognitive science, patter recognition, image processing, signal analysis and many other fields[274–276]. By the definition of rough sets, any subset of a universe can be characterized by two definable subsets which are called lower and upper approximations. However, the equivalence relations in Pawlak rough sets are too restrictive for theoretical and practical aspects, many researchers have generalized the concepts of Pawlak rough set by using non-equivalence relations. Some generalized rough set models can be found in [17,65,142,150,214,221,313,324–326,330,338,339,341,376–378]. It is worth noting that some researchers applied this theory to algebraic structures and obtained some interesting results. Biswas et al.[50] introduced the concept of rough subgroups. Furthermore, many new rough algebraic models have also been established. In [71], Davvaz studied the notions of rough ideals and rough subrings with respect to an ideal of a ring. Rough modules have been investigated by Davvaz et al.[76] Recently, Ali[18] investigated some properties of roughness in hemirings.

Soft set theory is free from the difficulties that have troubled the usual theoretical approaches. Afterwards, a wide range of applications of soft sets have been studied in many different fields including the smoothness of functions, game theory, operation researches, Riemann integrations, probability theory and measurement theory[241, 255]. In recent years, researches works on soft sets are very energetic and progressing quickly. In particular, Çağman[59, 60] applied soft theory to decision making. Based on fuzzy soft sets, Roy and Maji[285] raised a method of object recognition from an imprecise multi-observer data and applied it to decision making problems. Maji et al.[241] defined and researched several operations on soft sets. In 2009, Ali et al.[16, 18] gave some new operations on soft sets. Aktaş and Çağman[14] discussed relationships between soft sets and fuzzy sets as well as rough sets. They also defined soft groups and given some basic properties. Meanwhile, soft semirings and several related concepts are defined in [107]. In 2010, Sezer[295] put forward soft union set theory to rings. In particular, by combining Pawlak rough sets and soft sets, Feng et al.[110, 113] proposed the rough soft sets which can be regarded as a collection of rough sets sharing a common Pawlak approximation space.

In Chapter 1, we review some basic results on fuzzy sets, rough sets, soft sets and hemirings. The purpose of this chapter is to present basic results that are needed in the remainder of this book.

In Chapter 2, by means of a kind of new idea, we introduce certain fuzzy $h$-ideals ($(\in, \in\vee q)$-fuzzy $h$-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-ideals) of a hemiring, namely, fuzzy left (right) $h$-ideals ($(\in, \in\vee q)$-fuzzy $h$-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-ideals), fuzzy $h$-bi-ideals ($(\in, \in\vee q)$-fuzzy $h$-bi-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-bi-ideals), fuzzy $h$-quasi-ideals ($(\in, \in\vee q)$-fuzzy $h$-quasi-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-quasi-ideals) and fuzzy $h$-interior ideals($(\in, \in\vee q)$-fuzzy $h$-interior ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-interior ideals). We investigate their fundamental properties and mutual relationships.

In Chapter 3, we consider the characterizations of hemirings in the framework of fuzzy setting and ideal theory. The concepts of $h$-hemiregular hemirings, $h$-intra-hemiregular hemirings, $h$-quasi-hemiregular hemirings and $h$-semisimple hemirings are introduced. Then several characterizations of hemirings that are $h$-hemiregular and $h$-intra-hemiregular, $h$-hemiregular and $h$-intra-hemiregular, $h$-quasi-hemiregular, or $h$-semisimple are derived in terms of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy left (right) $h$-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-bi-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-quasi-ideals and $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy $h$-interior ideals of hemirings.

In Chapter 4, we initiate the study of soft hemirings by using the soft set theory. The notions of soft hemirings, soft subhemirings, soft ideals, idealistic soft hemirings and soft hemiring homomorphisms are introduced, and several related properties are investigated. Further, we investigate soft hemirings by the fuzzy theory. Some characterizations of hemirings are introduced by means of soft sets. Finally, we apply the concept of fuzzy soft sets to hemiring theory. The concepts of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left $h$-ideals (right $h$-ideals, $h$-bi-ideals, $h$-quasi-ideals) are introduced and some related properties are obtained. Some characterization theorems of $h$-hemiregular and $h$-semisimple hemirings are derived in terms of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft left (right) $h$-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft $h$-bi-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft $h$-quasi-ideals and $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy soft $h$-interior ideals.

In Chapter 5, we introduce certain $(M, N)$-$SI$-hemirings ($(M, N)$-$SI$-$h$-ideals) of hemirings, namely, $(M, N)$-$SI$-$h$-bi-ideals, $(M, N)$-$SI$-$h$-quasi-ideals, and then investigate their fundamental properties and mutual relationships.

In Chapter 6, we introduce certain $(M, N)$-$SU$-hemirings ($(M, N)$-$SU$-$h$-ideals) of hemirings, namely, $(M, N)$-$SU$-$h$-bi-ideals, $(M, N)$-$SU$-$h$-quasi-ideals. We investigate their fundamental properties and mutual relationships.

In Chapter 7, the concepts of $h$-hemiregular hemirings, $h$-intra-hemiregular hemirings and $h$-quasi-hemiregular hemirings are introduced. Then several characterizations of hemirings that are $h$-hemiregular and $h$-intra-hemiregular, $h$-hemiregular and $h$-intra-hemiregular or $h$-quasi-hemiregular are derived in terms of $(M, N)$-$SI$-$h$-ideals, $(M, N)$-$SI$-$h$-bi-ideals and $(M, N)$-$SI$-$h$-quasi-ideals of hemirings.

In Chapter 8, the concepts of $h$-hemiregular hemirings and $h$-intra-hemiregular hemirings are introduced. Then several characterizations of hemirings that are $h$-hemiregular and $h$-intra-hemiregular or $h$-hemiregular and $h$-intra-hemiregular are derived in terms of $(M, N)$-$SU$-$h$-ideals, $(M, N)$-$SU$-$h$-bi-ideals and $(M, N)$-$SU$-$h$-quasi-ideals of hemirings.

In Chapter 9, the purpose of this chapter is to introduce the concepts of fuzzy strong $h$-ideals and fuzzy congruences of hemirings. The quotient hemirings via fuzzy strong $h$-ideals are investigated. The relationships between fuzzy congruences and fuzzy strong $h$-ideals of hemirings are discussed. we also pay attention to an open question on fuzzy congruences. Further, by means of Dubois and Prade's idea, we apply rough fuzzy sets and fuzzy rough sets to algebraic structures. The concepts of rough fuzzy strong $h$-ideals and fuzzy rough strong $h$-ideals of hemirings are introduced, respectively. The relationships between them are investigated. Some important characterizations of these two kinds of rough set theory of hemirings are explored. Finally, we study roughness in soft hemirings with respect to Pawlak approximation spaces. Some new rough soft operations are explored.

The outcome of this book involves many people's efforts. First, I am very grateful to Prof. Young Bae Jun at Gyeongsang National University of Korea, Prof. Hee Sim Kim at Hanyang University of Korea, Prof. Bijan Davvaz at Yazd University of Iran, Prof. W.A. Dudek at Wroclaw University of Technology of Poland, Prof. Violeta-Elena Foeta at University Alexandru Toan Cuza of Romania, Prof. Irina Cristea at University of Nova Gorica of Slovenia, Prof. Piergiulio Corsini at Udine University of Italy, Prof. Baoding Liu at Tsinghua University of China, Profs. Bin Zhao, Yongming Li and Shenggang Li at Shanxi Normal University of China, Prof. Qingguo Li at Hunan University of China, Profs. Yang Xu and Keyun Qin at Southeast Jiaotong University of China, Prof. Fugui Shi at Beijing Institute of Technology of China, Profs. Hongxing Li and Xuehai Yuan at Dalian University of Technology of China, Prof. Baoqing Hu at Wuhan University of China, Prof. Xiaohong Zhang at Shanghai Maritime University of China, Prof. Feng Zhu at Minnan Normal University of China, Profs. Daowu Pei and Taihe Fan at Zhejiang Sci-Tec University of China, Prof. Xueping Wang at Sichuan Normal University of China, Prof. Yonglin Liu at Wuyi University of China, Prof. Xiaolong Xin at Northwest University of China, Prof. Wenyi Zeng at Beijing Normal University of China, Yunqiang Yin at Kunming University of Science and Technology of China, Prof. Feng Feng at Xi'an University of Posts and Telecommunications of China and so on. Second, my thanks are due to my Dr. Bin Yu for his excellent typesetting and editing of the manuscript with some helps from my colleagues and students Dajing Xiang, Xianping Liu, Min Zhou, Qi Liu, Kuanyun Zhu, Xiaowu Zhou and Wenjun Pan and so on. Third, we thank some doctors, such as Feng Qin, Hongjun Zhou, Pengfei He, Yongwei Yang, Jian Tang, Yanhong She and Xiaodong Pan and so on. Fourth, special thanks are also due to all the support in the preparation of the book by Hubei University for Nationalities of China. Fifth, we thank Key Subject of Mathematics of Hubei Province. Sixth, we gratefully acknowledge for grant support on our research by National Natural Science Foundation Committee of China (11461025). Finally, we thank the editors of Science Press for their kind consideration and advice to include this book.

Jianming Zhan
Jan. 2015.

# Contents

# Chapter 1
# Introduction

It is known that most of practical problems within the fields of economics, engineering, medical sciences, environmental sciences involve data that contain uncertainties. For this reason we can not successfully use traditional mathematical tools. In order to solve these problems, many scientists have put forth some special tools such as probability theory, fuzzy set theory [354], rough set theory [272, 273] and soft set theory [255].

## 1.1 Hemirings

A *semiring* is an algebraic system $(S,+,\cdot)$ consisting of a non-empty set $S$ together with two binary operations on $S$ called addition and multiplication (denoted in the usual manner) such that $(S,+)$ and $(S,\cdot)$ are semigroups and the following distributive laws

$$a\cdot(b+c)=a\cdot b+b\cdot c \text{ and } (a+b)\cdot c=a\cdot c+b\cdot c$$

are satisfied for all $a,b,c\in S$.

By zero of a semiring $(S,+,\cdot)$ we mean an element $0\in S$ such that $0\cdot x=x\cdot 0=0$ and $0+x=x+0=x$ for all $x\in S$. A semiring $(S,+,\cdot)$ with zero is called a *hemiring* if $(S,+)$ is commutative. For the sake of simplicity, we shall omit the symbol "$\cdot$", writing $ab$ for $a\cdot b\,(a,b\in S)$. If $A$ and $B$ are non-empty subsets in $S$, let $AB$ denote the set of all finite sums $\{a_1b_1+a_2b_2+\cdots+a_nb_n|n\in N, a_i\in A, b_i\in B\}$. Throughout this book, $S$ is always a hemiring.

A non-empty subset $A$ of $S$ is called a *left* (resp., *right*) *ideal* of $S$ if $A$ is closed under addition and $SA\subseteq A$ (resp., $AS\subseteq A$). Further, $A$ is called an *ideal* of $S$ if it is both a left ideal and a right ideal of $S$. A non-empty subset $B$ of $S$ is called a *bi-ideal* of $S$ if $B$ is closed under addition and multiplication satisfying $BSB\subseteq B$. A non-empty subset $Q$ of $S$ is called a *quasi-ideal* of $S$ if $Q$ is closed under addition and $SQ\bigcap QS\subseteq Q$. A subset $A$ of $S$ is called an *interior ideal* of $S$ if $A$ is closed under addition and multiplication such that $SAS\subseteq A$.

The *h-closure* $\overline{A}$ of $A$ in $S$ is defined as

$$\overline{A} = \{x \in S | x + a + z = b + z \text{ for some } a, b \in A, \ z \in S\}.$$

A subhemiring $A$ of $S$ is called a *strong h-subhemiring* if $x, y, z \in S, a, b \in A$ and $x+a+z = y+b+z$ implies $x \in y+A$. *Strong h-ideals* are defined similarly. Clearly, every strong $h$-subhemiring ($h$-ideal) is an $h$-subhemiring ($h$-ideal). A quasi-ideal $Q$ of $S$ is called an *h-quasi-ideal* of $S$ if $\overline{SQ} \cap \overline{SQ} \subseteq Q$ and $x + a + z = b + z$ implies $x \in Q$ for all $x, z \in S, a, b \in Q$.

**Lemma 1.1.1** For a hemiring $S$, we have

(1) $A \subseteq \overline{A}$, where $\{0\} \subseteq A \subseteq S$;

(2) If $A \subseteq B \subseteq S$, then $\overline{A} \subseteq \overline{B}$;

(3) $\overline{\overline{A}} = \overline{A}$, $\forall A \subseteq S$;

(4) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$;

(5) $\overline{ABC} = \overline{\overline{A}\,\overline{B}\,\overline{C}}$, $\forall A, B, C \subseteq S$;

(6) For any left (right) $h$-ideal, $h$-bi-ideal, $h$-quasi-ideal or $h$-interior ideal $A$ of $S$, we have $A = \overline{A}$.

**Proof** We only show (4), the other properties can be easily proved. Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, then $AB \subseteq \overline{A}\,\overline{B}$, and so $\overline{AB} \subseteq \overline{\overline{A}\,\overline{B}}$.

To show the converse inclusion, let $x \in \overline{A}$ and $y \in \overline{B}$. Then there exist $a_1, a_2 \in A, b_1, b_2 \in B$ and $z_1, z_2 \in S$ such that $x + a_1 + z_1 = a_2 + z_1$ and $y + b_1 + z_2 = b_2 + z_2$. Then we have

$$xy + a_1 y + z_1 y = a_2 y + z_1 y,$$

$$a_1 y + a_1 b_1 + a_1 z_2 = a_1 b_2 + a_1 z_2$$

and

$$a_2 y + a_2 b_1 + a_2 z_2 = a_2 b_2 + a_2 z_2.$$

Since $a_1 b_1, a_1 b_2, a_2 b_1$ and $a_2 b_2$ are elements of $AB$, we have

$$a_1 y + a_1 b_1 + a_1 z_2 = a_1 b_2 + a_1 z_2$$

and

$$a_2 y + a_2 b_1 + a_2 z_2 = a_2 b_2 + a_2 z_2,$$

imply $a_1 y, a_2 y \in \overline{AB}$. Thus $xy + a_1 y + z_1 y = a_2 y + z_1 y$ gives that $xy \in \overline{\overline{A}\,\overline{B}}$, that is, $\overline{\overline{A}\,\overline{B}} \subseteq \overline{AB}$. This completes the proof. □

A left ideal (resp., right ideal, ideal, bi-ideal, quasi-ideal, interior ideal) $A$ of $S$ is called a *left h-ideal* (resp., *right h-ideal, h-ideal, h-bi-ideal, h-quasi-ideal, h-interior ideal*) if $x, z \in S, a, b \in A$, and $x + a + z = b + z$ implies $x \in A$.

**Remark 1.1.2** (1) Every $h$-ideal ($h$-bi-ideal, $h$-quasi-ideal) of $S$ is an ideal (bi-ideal, quasi-ideal), respectively;

(2) Every $h$-ideal of $S$ is an $h$-quasi-ideal of $S$;

(3) Every $h$-quasi-ideal of $S$ is an $h$-bi-ideal of $S$;

(4) Every $h$-ideal of $S$ is an $h$-interior ideal of $S$.

However, the converse of the above remark does not hold in general as shown in the following examples.

**Example 1.1.3** Let $S = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

| $+$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ |
| $a$ | $a$ | $0$ | $b$ |
| $b$ | $b$ | $b$ | $0$ |

| $\cdot$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $b$ |

Then $S$ is a hemiring. Let $A = \{0, b\}$. Evidently $A$ is an ideal of $S$ and it is not an $h$-ideal of $S$, since $a + 0 + b = 0 + b$ while $a \notin A$.

**Example 1.1.4** Let $\mathbb{N}_0$ be the set of all non-negative integers and $S$ be the set of all $2 \times 2$ matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $(a_{ij} \in \mathbb{N}_0)$. Then $S$ is a hemiring with respect to the usual addition and multiplication of matrices. Consider the set $Q$ of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $(a \in \mathbb{N}_0)$. Evidently $Q$ is an $h$-quasi-ideal of $S$ and it is not a left (right) $h$-ideal of $S$.

**Example 1.1.5** We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of all positive integers and positive real numbers, respectively. The set $S$ of all matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a, b \in \mathbb{P}, c \in \mathbb{N})$ together with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a hemiring with respect to the usual addition and multiplication of matrices. Let $R$ and $L$ be the sets of all matrices $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a, b \in \mathbb{P}, c \in \mathbb{N}, a < b)$ together with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 0 \\ q & k \end{pmatrix}$ $(p, q \in \mathbb{P}, k \in \mathbb{N}, 3 < q)$ together with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, respectively. It is easy to show that $R$ and $L$ are a right $h$-ideal and a left $h$-ideal of $S$, respectively. Now the product $RL$ is an $h$-bi-ideal of $S$ and it is not an $h$-quasi-ideal of $S$. Indeed, the element

$$\begin{pmatrix} 6 & 0 \\ 9 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 3 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\right)$$
$$= \left(\begin{pmatrix} 1 & 0 \\ \frac{7}{6} & 1 \end{pmatrix}\begin{pmatrix} 24 & 0 \\ 4 & 1 \end{pmatrix}\right)\begin{pmatrix} \frac{1}{4} & 0 \\ 1 & 1 \end{pmatrix}$$

belongs to the intersection $\overline{S(RL)} \bigcap \overline{(RL)S}$, but it is not an element of $RL$. Hence $\overline{S(RL)} \bigcap \overline{(RL)S} \nsubseteq RL$.

**Example 1.1.6** Let $S = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\cdot$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $c$ | $c$ | $0$ |
| $b$ | $0$ | $c$ | $c$ | $0$ |
| $c$ | $0$ | $0$ | $0$ | $0$ |

Then $S$ is a hemiring. Let $I = \{0, a\}$. Evidently $I$ is an $h$-interior ideal of $S$ and it is not an $h$-ideal of $S$, since $a \in I, b \in S$ while $a \cdot b \notin I$.

**Definition 1.1.7** A hemiring $S$ is said to be *h-hemiregular* if for each $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$.

**Lemma 1.1.8** If $A$ and $B$ are, respectively, a right and a left $h$-ideal of $S$, then $\overline{AB} \subseteq A \bigcap B$.

**Proof** Let $x \in \overline{AB}$. Then $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$ for $a_i, a'_j \in A$, $b_i, b'_j \in B$ and $z \in S$. Since $A$ is a right $h$-ideal of $S$ and $(S, +)$ is a commutative semigroup, and $\sum_{i=1}^{m} a_i b_i$, $\sum_{j=1}^{n} a'_j b'_j \in A$, therefore $x \in A$. Similarly, we can prove that $x \in B$. So, $x \in A \bigcap B$, i.e., $\overline{AB} \subseteq A \bigcap B$. □

**Lemma 1.1.9** A hemiring $S$ is $h$-hemiregular if and only if for any right $h$-ideal $A$ and any left $h$-ideal $B$, we have $\overline{AB} = A \bigcap B$.

**Proof** Assume that $S$ is $h$-hemiregular and $a \in A \bigcap B$. Then there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$. Since $A$ is a right $h$-ideal of $S$, we have $ax_i \in A$ and $ax_ia \in AB$ for $i = 1, 2$. Thus $a \in \overline{AB}$, which implies $A \bigcap B \subseteq \overline{AB}$. By Lemma 1.1.8, $\overline{AB} = A \bigcap B$.

Conversely, let $a \in S$. It is not difficult to verify $aS + N_0a$ is the principal right ideal of $S$ generated by $a$, where $N_0 = \{0, 1, 2, \cdots\}$. Consequently, $\overline{(aS + N_0a)}$ is a right $h$-ideal of $S$. Therefore

$$\overline{(aS + N_0a)} = \overline{(aS + N_0a)} \bigcap S = \overline{\overline{(aS + N_0a)}S} = \overline{(aS + N_0a)S} = \overline{aS}$$

because $S$ is trivially an $h$-ideal of itself. Thus

$$a = a \cdot 0 + 1 \cdot a \in aS + N_0 a \subseteq \overline{(aS + N_0 a)} = \overline{aS}.$$

Similarly, $a \in \overline{Sa}$. Hence

$$a \in \overline{aS} \bigcap \overline{Sa} = \overline{\overline{aS} \cdot \overline{Sa}} = \overline{aSSa} \subseteq \overline{aSa},$$

since $\overline{aS}$ and $\overline{Sa}$ are, respectively, right and left $h$-ideal of $S$. This shows that there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$. So, $S$ is an $h$-hemiregular hemiring. □

**Lemma 1.1.10** Let $S$ be a hemiring. Then the following conditions are equivalent:

(1) $S$ is $h$-hemiregular;
(2) $B = \overline{BSB}$ for every $h$-bi-ideal $B$ of $S$;
(3) $Q = \overline{QSQ}$ for every $h$-quasi-ideal $Q$ of $S$.

**Proof** (1)⇒(2). Assume that (1) holds. Let $B$ be any $h$-bi-ideal of $S$ and $x$ any element of $B$. Then there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$. It is easy to see that $xax, xa'x \in BSB$ and so $x \in \overline{BSB}$. Hence $B \subseteq \overline{BSB}$. On the other hand, since $B$ is an $h$-bi-ideal of $S$, we have $BSB \subseteq B$ and so $\overline{BSB} \subseteq \overline{B} = B$ by Lemma 1.1.1. Therefore, $B = \overline{BSB}$.

(2)⇒(3). This is straightforward.

(3)⇒(1). Assume that (3) holds. Let $R$ and $L$ be any right $h$-ideal and any left $h$-ideal of $S$, respectively. Then we have

$$\overline{(R \bigcap L)S} \bigcap \overline{S(R \bigcap L)} \subseteq \overline{RS} \bigcap \overline{SL} \subseteq \overline{R} \bigcap \overline{L} = R \bigcap L,$$

and thus $R \bigcap L$ is an $h$-quasi-ideal of $S$. By the assumption and Lemma 1.1.1, we have

$$R \bigcap L = \overline{(R \bigcap L)S(R \bigcap L)} \subseteq \overline{RSL} \subseteq \overline{RL} \subseteq \overline{R \bigcap L} \subseteq \overline{R} \bigcap \overline{L} = R \bigcap L,$$

and thus $\overline{RL} = R \bigcap L$. Therefore, $S$ is $h$-hemiregular by Lemma 1.1.9. □

**Corollary 1.1.11** Let $S$ be a hemiring. Then the following conditions are equivalent:

(1) $S$ is $h$-hemiregular;
(2) $B \bigcap A = \overline{BAB}$ for every $h$-bi-ideal $B$ and every $h$-ideal $A$ of $S$;
(3) $Q \bigcap A = \overline{QAQ}$ for every $h$-quasi-ideal $Q$ and every $h$-ideal $A$ of $S$.

**Proof** The proof is similar to that of Lemma 1.1.10. □

**Corollary 1.1.12** Let $S$ be a hemiring. Then the following conditions are equivalent:

(1) $S$ is $h$-hemiregular;

(2) $B\bigcap L \subseteq \overline{BL}$ for every $h$-bi-ideal $B$ and every left $h$-ideal $L$ of $S$;

(3) $Q\bigcap L \subseteq \overline{QL}$ for every $h$-quasi-ideal $Q$ and every left $h$-ideal $L$ of $S$;

(4) $R\bigcap B \subseteq \overline{RB}$ for every right $h$-ideal $R$ and every $h$-bi-ideal $B$ of $S$;

(5) $R\bigcap Q \subseteq \overline{RQ}$ for every right $h$-ideal $R$ and every $h$-quasi-ideal $Q$ of $S$;

(6) $R\bigcap B\bigcap L \subseteq \overline{RBL}$ for every right $h$-ideal $R$, every $h$-bi-ideal $B$ and every left $h$-ideal $L$ of $S$;

(7) $R\bigcap Q\bigcap L \subseteq \overline{RQL}$ for every right $h$-ideal $R$, every $h$-quasi-ideal $Q$ and every left $h$-ideal $L$ of $S$.

**Proof** The proof is similar to that of Lemma 1.1.10. □

**Definition 1.1.13** A hemiring $S$ is said to be *$h$-intra-hemiregular* if for each $x \in S$, there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that $x + \sum\limits_{i=1}^{m} c_i x^2 d_i + z = \sum\limits_{j=1}^{n} c'_j x^2 d'_j + z$. Equivalent definitions: (1) $x \in \overline{Sx^2S}, \forall x \in S$; (2) $A \subseteq \overline{SA^2S}, \forall A \subseteq S$.

It is not difficult to observe that in any case of rings the $h$-intra-hemiregularity coincides with the classical intra-regularity of rings.

**Example 1.1.14** (1) The set $\mathbb{N}_0$ of all non-negative integers with usual addition and multiplication is a hemiring, but it is neither $h$-hemiregular nor $h$-intra-hemiregular. Indeed, the element $2 \in \mathbb{N}_0$ can not be expressed as $2+2a2+z = 2a'2+z$ or $2 + \sum\limits_{i=1}^{m} a_i 2^2 a'_i + z' = \sum\limits_{j=1}^{n} b_j 2^2 b'_j + z'$ for all $a, a', a_i, a'_i, b_j, b'_j, z, z' \in \mathbb{N}_0$;

(2) Let $S = \{0, a, b, c\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

| $+$ | 0 | $a$ | $b$ |
|---|---|---|---|
| 0 | 0 | $a$ | $b$ |
| $a$ | $a$ | $a$ | $b$ |
| $b$ | $b$ | $b$ | $b$ |

| $\cdot$ | 0 | $a$ | $b$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $a$ |
| $b$ | 0 | $a$ | $a$ |

Then $S$ is a hemiring that is both $h$-hemiregular and $h$-intra-hemiregular.

**Lemma 1.1.15** Let $S$ be a hemiring. Then the following conditions are equivalent:

(1) $S$ is $h$-intra-hemiregular;

(2) $L\bigcap R \subseteq \overline{LR}$ for every left $h$-ideal $L$ and every right $h$-ideal $R$ of $S$.

**Proof** (1) $\Rightarrow$ (2). Assume that (1) holds. Let $L$ and $R$ be any left $h$-ideal and any right $h$-ideal of $S$, respectively. Since $S$ is $h$-intra-hemiregular, we have

$$L\bigcap R\subseteq \overline{S(L\bigcap R)^2S}=\overline{(S(L\bigcap R))((L\bigcap R)S)}\subseteq \overline{(SL)(RS)}\subseteq \overline{LR}.$$

(2) $\Rightarrow$ (1). Assume that (2) holds. Let $a\in S$. Then it is easy to see that $\overline{Sx+Mx}$ and $\overline{xS+Nx}$ are the principal left $h$-ideal and principal right $h$-ideal of $S$ generated by $x$, respectively, where $M=\{0,1,2,\cdots\}$ and $N=\{0,1,2,\cdots\}$. By the assumption, we have

$$\begin{aligned}x=&x\cdot 0+1\cdot x\in \overline{Sx+Mx}\bigcap\overline{xS+Nx}\\ \subseteq&\overline{\overline{(Sx+Mx)}\,\overline{(xS+Nx)}}\\ =&\overline{(Sx+Mx)(xS+Nx)}\\ =&\overline{(Sx)(xS)+(Sx)(Nx)+(Mx)(xS)+(Mx)(Nx)}.\end{aligned}$$

Thus we have

$$x+\sum_{i=1}^{m}a_ix^2a_i'+z=\sum_{j=1}^{n}b_jx^2b_j'+z$$

for some $a_i,a_i',b_j,b_j',z\in S$. This implies that $S$ is $h$-intra-hemiregular. □

**Lemma 1.1.16** Let $S$ be a hemiring. Then the following are equivalent:
(1) $S$ is both $h$-hemiregular and $h$-intra-hemiregular;
(2) $B=\overline{B^2}$ for every $h$-bi-ideal $B$ of $S$;
(3) $Q=\overline{Q^2}$ for every $h$-quasi-ideal $Q$ of $S$.

**Proof** (1) $\Rightarrow$ (2). Assume that (1) holds. Let $B$ be any $h$-bi-ideal of $S$ and $x$ any element of $B$. Then $\overline{B^2}\subseteq\overline{B}=B$. Since $S$ is both $h$-hemiregular and $h$-intra-hemiregular, there exist some elements $a_1,a_2,p_i,p_i',q_j,q_j',z_1$ and $z_2$ of $S$ such that

$$x+xa_1x+z_1=xa_2x+z_1 \tag{1.1.1}$$

and

$$x+\sum_{i=1}^{m}p_ix^2p_i'+z_2=\sum_{j=1}^{n}q_jx^2q_j'+z_2. \tag{1.1.2}$$

Then by (1.1.1) we have

$$xa_1x+xa_1xa_1x+z_1a_1x=xa_2xa_1x+z_1a_1x \tag{1.1.3}$$

and

$$xa_2x+xa_1xa_2x+z_1a_2x=xa_2xa_2x+z_1a_2x. \tag{1.1.4}$$

Combining (1.1.1), (1.1.3) and (1.1.4), we have

$$\begin{aligned}&x+xa_1x+z_1+xa_1xa_1x+xa_1xa_2x+z_1a_1x+z_1a_2x\\ =&xa_2x+z_1+xa_1xa_1x+xa_1xa_2x+z_1a_1x+z_1a_2x,\end{aligned}$$