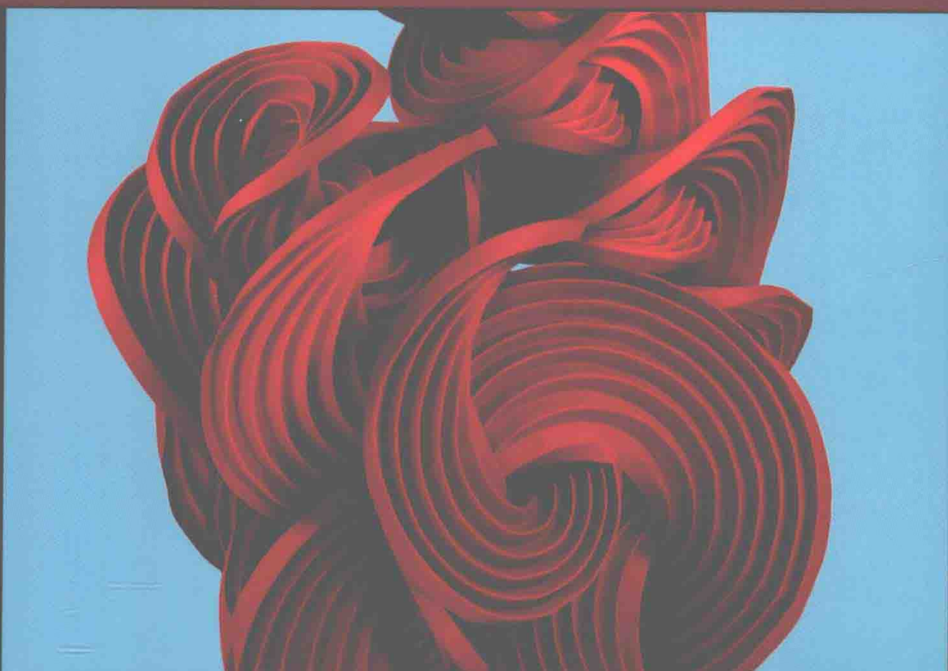
Volume 10

What's Happening in the Mathematical Sciences



DANA MACKENZIE
BARRY CIPRA

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BARRY CIPRA is a freelance mathematics writer based in Northfield, Minnesota. He received the 1991 Merten M. Hasse Prize from the Mathematical Association of America for an expository article on the Ising model, published in the December 1987 issue of the *American Mathematical Monthly*, and the 2005 Communication Award of the Joint Policy Board for Mathematics. His book, *Misteaks ... and how to find them before the teacher does...* (a calculus supplement), is published by AK Peters, Ltd.

About the Cover Images

Main image: Curved folds represent both a departure from tradition in origami and an area of active research from the artists, Erik Demaine and Martin Demaine. Below main image, left to right: A self-folding robot from Wyss Institute at Harvard University; a climate network developed at the Potsdam Institute for Climate Impact Research (PIK); Sherlock Holmes and John Watson, a line drawing by Sidney Paget, 1892; and coral in the Caribbean Sea overgrown by algae, an example of a climate "tipping point," from Craig Quirolo, Reef Relief. Back cover image: If each corner in the above path represents a city on a map, then this path is the shortest closed curve joining all the cities. Thus it solves the so-called "Traveling Salesman Problem" (TSP) for this particular map, from Robert Bosch.

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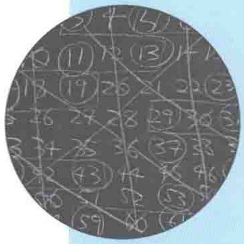
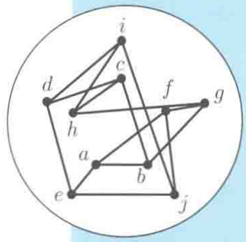
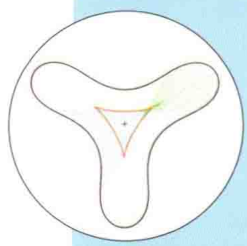






Origami: Unfolding the Future

Introduction

SEVERAL IMPORTANT RECENT DEVELOPMENTS in pure mathematics are featured in this volume of *What's Happening in the Mathematical Sciences.* **"Prime Clusters and Gaps: Out-Experting the Experts,"** page 18, talks about new insights into the distribution of prime numbers, the perpetual source of new problems, and new results. Recently, several mathematicians (including Yitang Zhang and James Maynard) significantly improved our knowledge of the distribution of prime numbers. Advances in the so-called Kadison-Singer problem and its applications in signal processing algorithms used to analyze and synthesize signals are described in **"The Kadison-Singer Problem: A Fine Balance,"** page 72. **"Quod Erat Demonstrandum,"** page 64, presents two examples of perseverance in mathematicians' pursuit of truth using, in particular, computers to verify their arguments. Also, **"Following in Sherlock Holmes' Bike Tracks,"** page 52, shows how an episode in one of Sir Arthur Conan Doyle's stories about Sherlock Holmes naturally led to very interesting problems in the theory of completely integrable systems.

On the applied side, **"Climate Past, Present, and Future,"** page 36, shows the importance of mathematics in the study of problems of climate change and global warming. Mathematical models help researchers to understand the past, present, and future changes of climate, and to analyze their consequences. Economists have known for a long time that trust is a cornerstone of commerce. **"The Truth Shall Set Your Fee,"** page 28, shows how recent advances in theoretical computer science led to the development of so-called "rational protocols" for information exchange, where the seller of information is forced to tell the truth in order to maximize profit.

Over the last 100 years many professional mathematicians and devoted amateurs contributed to the problem of finding polygons that can tile the plane, e.g., used as floor tiles in large rooms and walls. Despite all of these efforts, the search is not yet complete, as the very recent discovery of a new plane-tiling pentagon shows in **"A Pentagonal Search Pays Off,"** page 86. The increased ability to collect and process statistics, big data, or "analytics" has completely changed the world of sports analytics as shown in **"The Brave New World of Sports Analytics,"** page 96. The use of modern methods of statistical modeling allows coaches and players to create much more detailed game plans in professional baseball and basketball as well as create many new ways of measuring a player's value. Finally, **"Origami: Unfolding the Future,"** page 2, talks about the ancient Japanese paper-folding art and origami's unexpected connections to a variety of areas including mathematics, technology, and education.

Fire Tower (2013). *Curved folds represent both a departure from tradition in origami and an area of active research. (Courtesy of the artists, Erik Demaine and Martin Demaine.)*

Origami: Unfolding the Future

Dana Mackenzie

ELEGANT IN ITS DESIGN and simple in its materials, origami (or paper-folding) has for more than 1400 years been a quintessential Japanese art. Generations of children and adults have learned how to fold classical figures like the crane, the jumping frog, and the flapping bird.

In the last few years, however, origami has taken off in a new, high-tech direction. Engineers are now designing retinal implants that will unfold inside the human eye, and enable people who have lost their vision to see again; robots that fold themselves out of a flat plane into three dimensions and then walk away; and "metamaterials" that change their physical properties on demand. Origami engineering uses new materials, breaks size barriers (creating objects that are scarcely larger than a mote of dust), and places a new emphasis on functionality rather than aesthetics. It also poses some new mathematical challenges.

Much of the explosion in research can be attributed to a visionary program of the National Science Foundation, called Origami Design for Integration of Self-assembling Systems for Engineering Innovation, or ODISSEI. Conceived by Glaucio Paulino of NSF's Engineering Directorate, ODISSEI funded eight projects in 2012 and five more in 2013, each one to the tune of \$2 million. (The Air Force's Office of Scientific Research also kicked in some of the money.) "The ODISSEI program has made a huge difference," says Larry Howell of Brigham Young University, principal investigator on one of the grants. "As an engineer, when you do something as bold and audacious as origami, people ask you what you are talking about. You show them the funding source and it gives you instant credibility."

Some History of Mathematical Origami

Even before ODISSEI, origami science had been quietly gathering momentum for more than two decades. One of its pioneers in the United States was Robert Lang, a former engineer at the Jet Propulsion Laboratory and lifelong origami artist. In the early 1990s, Lang developed the first computer program for designing origami models, called TreeMaker. If you wanted to make an origami model for anything—say a beetle or a rhinoceros—and you could draw a stick figure of it, TreeMaker could calculate the folds required to bring your figure to life. (See Figure 1, next page.)

TreeMaker works by computing a circle packing of a square sheet of paper, which allocates each disk to make one appendage of the future model. The sizes of the disks correspond roughly to the sizes of the appendages, and adjacent disks correspond to adjacent appendages. Ironically, Lang says that

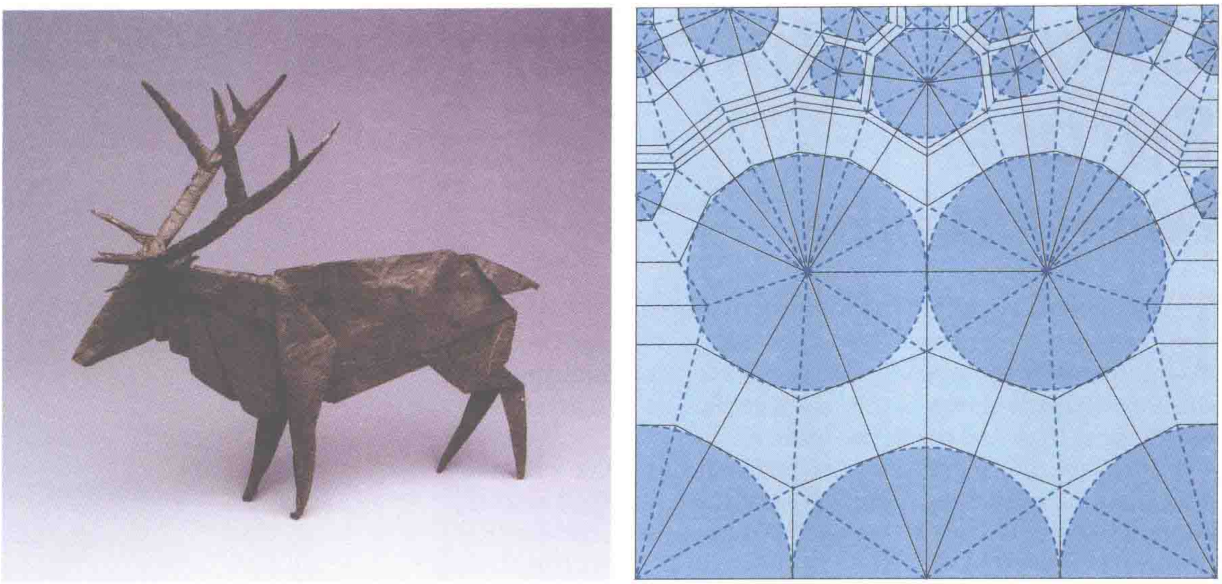


Figure 1. *"Roosevelt Elk, opus 358." It is believed that Robert Lang's "TreeMaker" software can print out a fold pattern to create an origami version of any 3-dimensional object, provided that the object can be drawn as a stick figure. However, this universality property has not been formally proved. (Left) An elk. (Right) The fold pattern used to create it. (Courtesy of Robert J. Lang, http://langorigami.com.)*

he seldom uses TreeMaker in his own art. "TreeMaker is a single idea, and I like to combine multiple concepts," he explains.

In 1998, Erik Demaine of the Massachusetts Institute of Technology (together with his father, Martin Demaine, and Joseph Mitchell of Stony Brook University) proved the foldability of *any* polygonal silhouette and *any* three-dimensional polyhedron from a single square of paper. In other words, if you can visualize it, you can fold it. However, the piece of paper may need to be enormous. As Demaine says, it would be interesting to know an efficient universal folding algorithm. The closest thing to date is another algorithm called Origamizer, written by Tomohiro Tachi of the University of Tokyo in 2008, for which Tachi and Demaine are close to proving universality.

Japan, the birthplace of origami, began to awaken to the potential of mathematical origami around the same time as people like Lang and Demaine in the West. Jun Maekawa, a physicist and software engineer, pioneered the study of crease patterns (the network of "mountain folds" and "valley folds" left in the paper when it has been folded into a model and unfolded again).

One of the most frequently cited theorems of origami mathematics is named after Maekawa. In classical origami, the final goal is usually an animal, such as a rhinoceros or a beetle, or an everyday object in three dimensions. But in many applications of origami, one needs to fold an initially large amount of material into a very small space. If the material could be folded flat, its volume would be essentially zero. (In the real world, of course, materials have thickness and volumes are not zero—but at least this is a target the designer can aim for.) So the question is: Which crease patterns can fold up into a flat, two-dimensional figure? Maekawa's theorem gives a necessary condition: at any flat-foldable vertex, the number of mountain folds and the number of valley folds must differ by two.

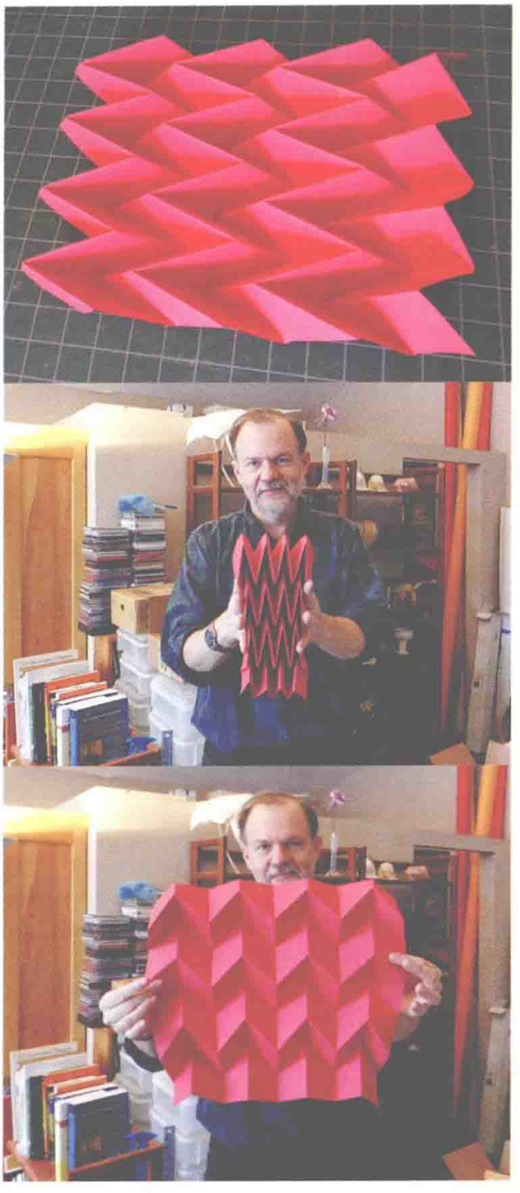
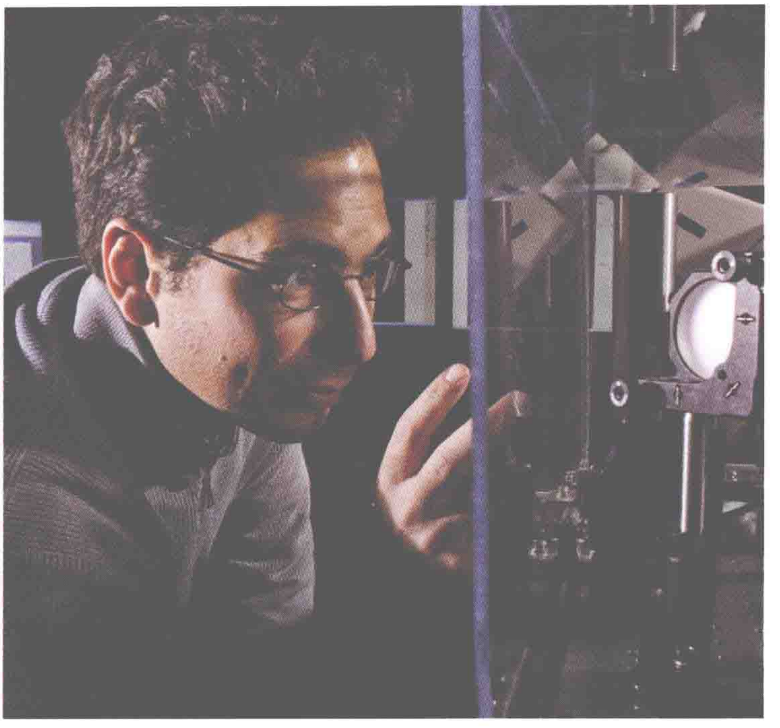


Figure 2.*Robert Lang with Miura-ori fold. (Top) Semi-folded Miura-ori. (Center) Miura-ori in compressed state. (Bottom) Miura-ori fully extended. The transition from folded to unfolded is accomplished by pulling the sides out in one smooth motion. (Photo courtesy of Dana Mackenzie.)*

Koryo Miura of the Japanese Institute of Space and Astronautical Science (ISAS) made another hugely important discovery about flat folding. As early as the 1970s, Miura began thinking about the problem of how a thin, flexible plate will buckle if you apply uniform compression to the outside. He discovered a prototypical solution, a sort of herringbone pattern, now called the Miura-ori or "Miura fold." The unfolded crease pattern for the Miura-ori looks like a pattern of zigzagging parallelograms. In accordance with Maekawa's theorem, there are three mountain



Itai Cohen. *(Photo courtesy of Cornell University Photography.)*

folds and one valley fold at every vertex, or vice versa. While tricky to fold from scratch, the Miura-ori is incredibly easy to fold when the paper is pre-scored. All you have to do is grab two corners and pull out; to fold it back in again, you take two corners and push in. (See Figure 2, previous page.)

These motions will cause the entire pattern, no matter how large, to expand or collapse at once. The Miura-ori is the ultimate solution to the map-folding problem. When you try to fold a conventional map, it nearly always folds up wrong. There are too many folds, and inevitably some of them will be done in the wrong order or in the wrong direction. With a Miura-ori map, on the other hand, folding or unfolding is foolproof, and it only takes a second.

It is often said that Miura developed his fold for the Japanese space program. That is not quite accurate, because his work started as the solution to a theoretical math problem. However, a solar array folded in the Miura-ori pattern flew aboard ISAS' Space Flyer Unit, launched in 1995, which demonstrated for the first time the practicality of using origami to stow a large array of solar panels inside a small spaceship.

Pop Goes the Defect

The Miura-ori is fundamental for anyone who wants to understand how movable origami works. Itai Cohen, a physicist at Cornell University, calls it "the hydrogen atom of origami," because it is one of the simplest possible tessellated folding patterns, yet still exhibits a variety of interesting and unexpected behaviors.

One remarkable feature of the Miura-ori crease pattern is its near-rigidity: it has one degree of freedom. A rigid object has no degrees of freedom. The one degree of freedom of the Miura-ori is provided by pulling out or pushing in on the corners. Once the motion is initiated in one place, the whole object has no choice but to follow. It has no unwanted, or "parasitic" motions.

The Miura-ori also has a remarkable physical property called a *negative Poisson ratio.* (Materials with this property are sometimes called *auxetic*, which is a little bit easier to say.) If you stretch most materials in one direction, they will tend to shrink in the transverse direction(s). The ratio between the amount of stretching in one direction and the amount of shrinking in the other(s) is called the Poisson ratio, and it is normally between 0 (e.g., cork) and 1/2 (e.g., rubber).

However, when you stretch a material with a negative Poisson ratio in one direction, it will stretch in the transverse direction(s) as well. The Miura-ori behaves exactly like this: pull the sides of the map, and the top and bottom will move out at the same time.

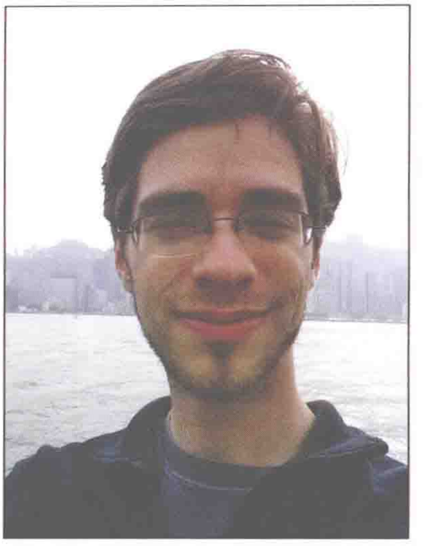
The physicist who is widely credited with inventing the first synthetic auxetic material was Rod Lakes of the University of Wisconsin. He described his material, a polymer foam with nonconvex unit cells, in an article in *Science* in 1987. By now, many auxetic materials have been created. They have found application in sports wear, for instance, because they are good shock absorbers.

In 2010, Jesse Silverberg, a lifetime origami enthusiast and graduate student in physics at Cornell, happened to go out to dinner after a talk with Chris Santangelo, a professor, and Marcelo Diaz, a fellow graduate student. The talk was about the geometry of folding so, Silverberg says, "I started folding something I had learned years and years ago"—namely, the Miura-ori. "I passed it to Marcelo and he passed it to Chris, and each of them said, holy crap!" What they had in front of them was an auxetic material—only it wasn't made of polymer foam, it was made out of simple paper. And it had been known long before 1987. They could be forgiven their holy-crap moment; it was like meeting someone from a different planet, and finding out that he already knows about the most sophisticated discoveries on your own planet.

Technically, the Miura-ori folded paper is not a material but a metamaterial: a material whose physical properties are partly based on its shape or configuration. An even simpler example of a metamaterial is reinforced cardboard. Three pieces of paper, stacked on top of each other, are not very stiff. But if you make the inside layer corrugated, the three layers together become stiffer and you have something that you can make boxes out of.

When the ODISSEI program was announced, Santangelo and Cohen sent in a proposal to study origami-based metamaterials, and they were funded in 2012. As it turned out, the surprises had just begun.

One of their collaborators, mathematician Thomas Hull of Western New England University, was particularly interested in misfolded Miura-ori (see Figure 3, next page). This may seem like a curious preoccupation, given the fact that the whole advantage of Miura-ori is the ease of folding it correctly. Nevertheless, Hull had noticed that if you deliberately convert some of the mountain folds in the Miura-ori to valleys and vice versa, you can sometimes get a flat-foldable pattern. He set his student Jessica Ginepro to work on counting the number of misfolded patterns in a 2-by-n Miura-ori, a 3-by-n, and so on.



Jesse Silverberg. *(Photo courtesy of Jesse L. Silverberg.)*

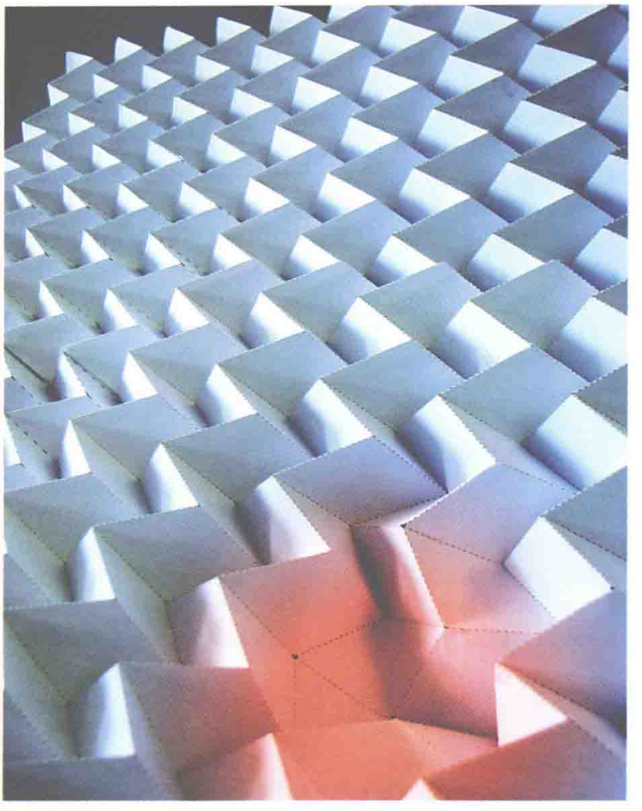


Figure 3. *Misfolded Miura-ori. The orange vertex has been "popped" downward, creating a defect. (Figure courtesy of Jesse L. Silverberg, Arthur A. Evans, Lauren McLeod, Ryan C. Hayward, Thomas Hull, Christian D. Santangelo, and Itai Cohen.)*

Meanwhile, Santangelo, Cohen and Silverberg were looking at another way of mutilating a Miura-ori, by "popping through" its corners. If you push on one vertex of a Miura-ori, you can get it to pop through, converting a peak to a sink or vice versa. The popped Miura-ori is no longer flat foldable; the faces next to the popped vertex have to be bent. (The mere fact that it "pops" indicates that it is not a rigid motion. A rigid door swinging on a hinge does not pop.)

The first discovery that they made was that a Miura-ori with popped vertices is much stiffer than a non-popped one (see Figure 4). Maekawa's theorem mathematically forbids you to fold it flat, because the popping changes the number of mountain folds and valley folds at each vertex. Physically, if you try to fold it flat, its resistance to compression will skyrocket until one of the creases tears. "With paper, you can get it 10 to 100 times stiffer by introducing defects," Silverberg says. "Ten is what we were able to easily achieve, and around 100 we run into a limit because of the strength of the material. If we worked with plastic sheets, or something with a higher resistance to tearing, then 100 could be just the beginning."

The variable-stiffness property of popped Miura-ori makes it a tunable metamaterial. Unlike corrugated cardboard, whose stiffness is set once and for all, you can adjust the stiffness of

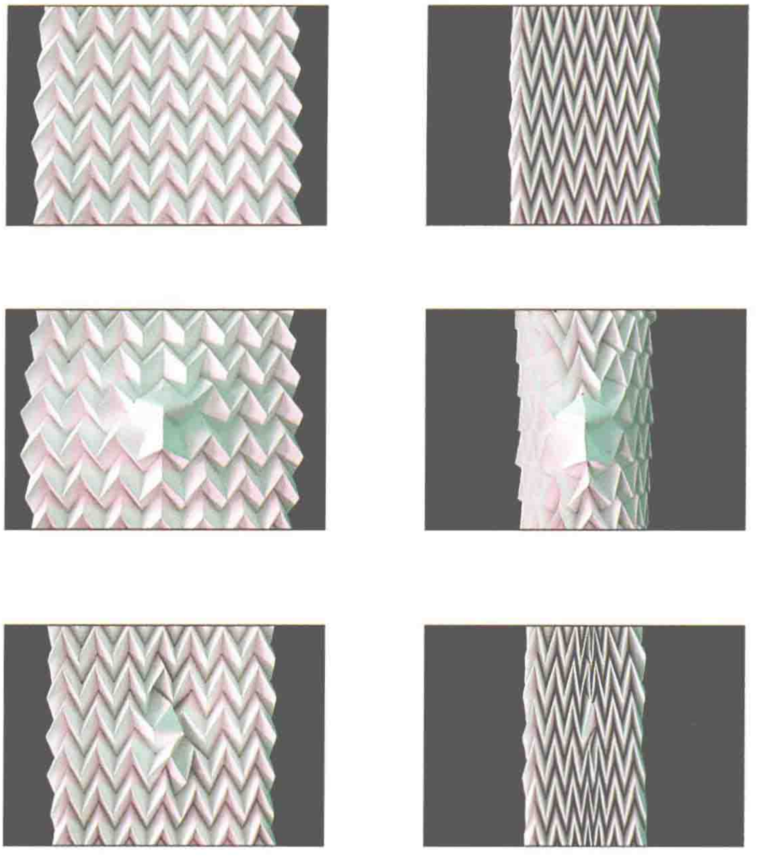
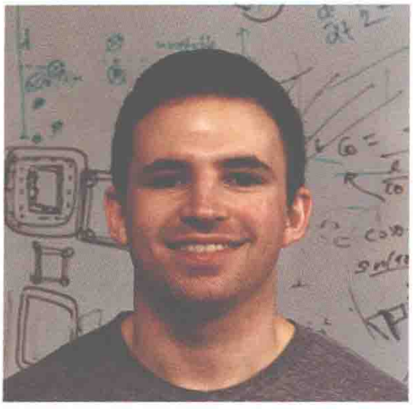


Figure 4. *(Top) Properly folded Miura-ori collapses easily. (Center) Defect in a misfolded Miura-ori resists collapsing. (Bottom) Pairs of defects can sometimes "cancel out" and create a flat-foldable design. (Figure courtesy of Jesse L. Silverberg, Arthur A. Evans, Lauren McLeod, Ryan C. Hayward, Thomas Hull, Christian D. Santangelo, and Itai Cohen.)*

a Miura-ori by increasing or decreasing the density of defects. You can also arrange the defects to make a hinge, or vice versa, to make a strut that is exceptionally resistant to bending. "Our idea is to use self-folding techniques to create robot limbs that change their mechanical properties," says Cohen. "You can fold a claw around something, make the claw rigid, and then pick it up."

Perhaps the most surprising discovery was the fact that defects can cancel each other out and make the Miura-ori flat-foldable again. The team realized that some of Hull's misfolded Miura-ori had a crease pattern that looked like two adjacent popped-through vertices. "Our spidey sense was tingling," Silverberg says. In fact, the equal-and-opposite pops restore the numbers of valley and mountain folds so that they once again obey Maekawa's theorem. Once the mathematical obstacle to flat foldability disappears, the increased stiffness also vanishes. "The idea of defects interacting came out of nowhere," says Cohen. "It was a total surprise. I had some initial intuition about the defects, but it wasn't until Jesse started working with the models that we discovered this. It was not what I predicted—it was much cooler."



Sam Felton. *(Photo courtesy of Peter York.)*

Self-Assembling Robots

The Cornell group reported their results on tunable metamaterials in *Science* magazine in August 2014. Not so coincidentally, their article appeared back-to-back with an article by another ODISSEI-funded group, co-organized by Demaine, Daniela Rus of MIT and Robert Wood of Harvard University. This paper, written by graduate student Sam Felton, reported on the first self-folding and self-activating robot (see Figure 5).

The Harvard/MIT group in fact progressed toward the robot in several stages. In earlier steps, they built a lamp that assembled itself but couldn't turn itself on, and an inchworm that could move on its own but required some human help in assembly. At the same time they were experimenting with several different shape-memory materials. According to Felton, their "secret sauce" is something that can be bought in a toy store—a heat-activated polymer called prestressed polystyrene or Shrinky Dinks. When heated to about 100 degrees Celsius, the Shrinky Dinks contract.

If a line is cut in a piece of cardboard and a piece of polystyrene is taped over it, the polystyrene acts as a hinge. As it contracts, it pulls the two sides of cardboard toward one another, creating a fold. The change is permanent, because the Shrinky Dinks cannot expand again after they have contracted. The angle of the fold can be controlled by the thickness and spacing of the two pieces of cardboard. Somewhat ironically for a project inspired by origami, the cardboard itself does not bend; it mostly adds stiffness and stability to the robot. The self-folding robot was made from five layers of material. On the outside were two layers of Shrinky Dinks (to allow both mountain folds and valley folds). Just inside them were two layers of cardboard, cut along each of the future folds. Finally, in the center of the sandwich was a single thin film that contains all the electronic wiring.

Felton first laid out the five-layer sandwich, with all the layers appropriately pre-scored and wired, and connected it to batteries and a microcontroller on top. This part of the assembly was done by hand, and took two hours. Then he turned the switch on. The microcontroller would execute a series of commands to heat up each wire in turn, activating the folds and causing the whole robot to rise up from the table like a salamander from the primordial ooze. This self-assembly phase took less than five minutes. After that, another series of pre-programmed commands caused the newly formed legs to move, and the robot trundled away under its own power. At a clip of 3 meters a minute, or about an eighth of a mile per hour, it couldn't win the Olympics but it could win a race with a snail.

Of course the self-folding robot was a proof of principle rather than a practical device. However, it has potential for application in two realms that pose challenges for traditional methods of assembly. One would be space missions, where it would be advantageous to store a rover as a flat panel in flight, then unfold and deploy it after arrival. With no humans on board, the rover would have to be self-assembling and self-activating.

A second application would be microscopic devices. For such devices, conventional assembly methods don't work because, as Cohen says, "We don't have good nano-screws yet." Three-dimensional printing, though it is all the rage in techie circles,