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组合数学

（英文版 · 第5版）

Introductory Combinatorics
Fifth Edition

Richard A. Brualdi

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出版者的话

文艺复兴以降，源远流长的科学精神和逐步形成的学术规范，使西方国家在自然科学的各个领域取得了垄断性的优势；也正是这样的传统，使美国在信息技术发展的六十多年间名家辈出、独领风骚。在商业化的进程中，美国的产业界与教育界越来越紧密地结合，计算机学科中的许多泰山北斗同时身处科研和教学的最前线，由此而产生的经典科学著作，不仅擘划了研究的范畴，还揭示了学术的源变，既遵循学术规范，又自有学者个性，其价值并不会因年月的流逝而减退。

近年，在全球信息化大潮的推动下，我国的计算机产业发展迅猛，对专业人才的需求日益迫切。这对计算机教育界和出版界都既是机遇，也是挑战；而专业教材的建设在教育战略上显得举足轻重。在我国信息技术发展时间较短的现状下，美国等发达国家在其计算机科学发展的几十年间积淀和发展的经典教材仍有许多值得借鉴之处。因此，引进一批国外优秀计算机教材将对我国计算机教育事业的发展起到积极的推动作用，也是与世界接轨、建设真正的世界一流大学的必由之路。

机械工业出版社华章分社较早意识到“出版要为教育服务”。自1998年开始，华章分社就将工作重点放在了遴选、移译国外优秀教材上。经过多年的不懈努力，我们与Pearson，McGraw-Hill，Elsevier，MIT，John Wiley & Sons，Cengage等世界著名出版公司建立了良好的合作关系，从他们现有的数百种教材中甄选出Andrew S. Tanenbaum，Bjarne Stroustrup，Brain W. Kernighan，Dennis Ritchie，Jim Gray，Afred V. Aho，John E. Hopcroft，Jeffrey D. Ullman，Abraham Silberschatz，William Stallings，Donald E. Knuth，John L. Hennessy，Larry L. Peterson等大师名家的一批经典作品，以“计算机科学丛书”为总称出版，供读者学习、研究及珍藏。大理石纹理的封面，也正体现了这套丛书的品位和格调。

“计算机科学丛书”的出版工作得到了国内外学者的鼎力襄助，国内的专家不仅提供了中肯的选题指导，还不辞劳苦地担任了翻译和审校的工

作；而原书的作者也相当关注其作品在中国的传播，有的还专程为其书的中译本作序。迄今，“计算机科学丛书”已经出版了近两百个品种，这些书籍在读者中树立了良好的口碑，并被许多高校采用为正式教材和参考书籍。其影印版“经典原版书库”作为姊妹篇也被越来越多实施双语教学的学校所采用。

权威的作者、经典的教材、一流的译者、严格的审校、精细的编辑，这些因素使我们的图书有了质量的保证。随着计算机科学与技术专业学科建设的不断完善和教材改革的逐渐深化，教育界对国外计算机教材的需求和应用都将步入一个新的阶段，我们的目标是尽善尽美，而反馈的意见正是我们达到这一终极目标的重要帮助。华章分社欢迎老师和读者对我们的工作提出建议或给予指正，我们的联系方法如下：

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Preface

I have made some substantial changes in this new edition of *Introductory Combinatorics*, and they are summarized as follows:

In Chapter 1, a new section (Section 1.6) on mutually overlapping circles has been added to illustrate some of the counting techniques in later chapters. Previously the content of this section occured in Chapter 7.

The old section on cutting a cube in Chapter 1 has been deleted, but the content appears as an exercise.

Chapter 2 in the previous edition (The Pigeonhole Principle) has become Chapter 3. Chapter 3 in the previous edition, on permutations and combinations, is now Chapter 2. Pascal's formula, which in the previous edition first appeared in Chapter 5, is now in Chapter 2. In addition, we have de-emphasized the use of the term *combination* as it applies to a set, using the essentially equivalent term of *subset* for clarity. However, in the case of multisets, we continue to use *combination* instead of, to our mind, the more cumbersome term *submultiset*.

Chapter 2 now contains a short section (Section 3.6) on finite probability.

Chapter 3 now contains a proof of Ramsey's theorem in the case of pairs.

Some of the biggest changes occur in Chapter 7, in which generating functions and exponential generating functions have been moved to earlier in the chapter (Sections 7.2 and 7.3) and have become more central.

The section on partition numbers (Section 8.3) has been expanded.

Chapter 9 in the previous edition, on matchings in bipartite graphs, has undergone a major change. It is now an interlude chapter (Chapter 9) on systems of distinct representatives (SDRs)—the marriage and stable marriage problems—and the discussion on bipartite graphs has been removed.

As a result of the change in Chapter 9, in the introductory chapter on graph theory (Chapter 11), there is no longer the assumption that bipartite graphs have been discussed previously.

The chapter on more topics of graph theory (Chapter 13 in the previous edition) has been moved to Chapter 12. A new section on the matching number of a graph (Section 12.5) has been added in which the basic SDR result of Chapter 9 is applied to bipartite graphs.

The chapter on digraphs and networks (Chapter 12 in the previous edition) is now Chapter 13. It contains a new section that revisits matchings in bipartite graphs, some of which appeared in Chapter 9 in the previous edition.

In addition to the changes just outlined, for this fifth edition, I have corrected all of the typos that were brought to my attention; included some small additions; made some clarifying changes in exposition throughout; and added many new exercises. There are now 700 exercises in this fifth edition.

Based on comments I have received over the years from many people, this book seems to have passed the test of time. As a result I always hesitate to make too many changes or to add too many new topics. I don't like books that have "too many words" (and this preface will not have too many words) and that try to accomodate everyone's personal preferences on topics. Nevertheless, I did make the substantial changes described previously because I was convinced they would improve the book.

As with all previous editions, this book can be used for either a one– or two–semester undergraduate course. A first semester could emphasize counting, and a second semester could emphasize graph theory and designs. This book would also work well for a one–semester course that does some counting and graph theory, or some counting and design theory, or whatever combination one chooses. A brief commentary on each of the chapters and their interrelation follows.

Chapter 1 is an introductory chapter; I usually select just one or two topics from it and spend at most two classes on this chapter. Chapter 2, on permutations and combinations, should be covered in its entirety. Chapter 3, on the pigeonhole principle, should be discussed at least in abbreviated form. But note that no use is made later of some of the more difficult applications of the pigeonhole principle and of the section on Ramsey's theorem. Chapters 4 to 8 are primarily concerned with counting techniques and properties of some of the resulting counting sequences. They should be covered in sequence. Chapter 4 is about schemes for generating permutations and combinations and includes an introduction to partial orders and equivalence relations in Section 4.5. I think one should at least discuss equivalence relations, since they are so ubiquitous in mathematics. Except for the section on partially ordered sets (Section 5.7) in Chapter 5, chapters beyond Chapter 4 are essentially independent of Chapter 4, and so this chapter can either be omitted or abbreviated. And one can decide not to cover partially ordered sets at all. I have split up the material on partially ordered sets into two sections (Sections 4.5 and 5.7) in order to give students a little time to absorb some of the concepts. Chapter 5 is on properties of the binomial coefficients, and Chapter 6 covers the inclusion–exclusion principle. The section on Möbius inversion, generalizing the inclusion–exclusion principle, is not used in later sections. Chapter 7 is a long chapter on generating functions and solutions of recurrence relations. Chapter 8 is concerned mainly with the Catalan numbers, the Stirling numbers of the first and second kind, partition numbers and the large and small Schröder numbers. One could stop at the end of any section of this chapter. The chapters that follow Chapter 8 are independent of it. Chapter 9 is about systems of distinct representatives (so-called marriage problems). Chapters 12 and 13 make some use of Chapter 9, as does the section on Latin squares in Chapter 10. Chapter 10 concerns some aspects of the vast theory of combinatorial designs and is independent of the remainder of the book. Chapters 11 and 12 contain an extensive discussion of graphs, with some emphasis on graph algorithms. Chapter 13 is concerned with digraphs and network flows. Chapter 14 deals with counting in the presence of the action of a permutation group and does make use of many of the earlier counting ideas. Except for the last example, it is independent of the chapters on graph theory and designs.

When I teach a one-semester course out of this book, I like to conclude with Burnside's theorem, and several applications of it, in Chapter 14. This result enables one to solve many counting problems that can't be touched with the techniques of earlier chapters. Usually, I don't get to Pólya's theorem.

Following Chapter 14, I give solutions and hints for some of the 700 exercises in the book. A few of the exercises have a $*$ symbol beside them, indicating that they are quite challenging. The end of a proof and the end of an example are indicated by writing the symbol $\Box$.

It is difficult to assess the prerequisites for this book. As with all books intended as textbooks, having highly motivated and interested students helps, as does the enthusiasm of the instructor. Perhaps the prerequisites can be best described as the mathematical maturity achieved by the successful completion of the calculus sequence and an elementary course on linear algebra. Use of calculus is minimal, and the references to linear algebra are few and should not cause any problem to those not familiar with it.

It is especially gratifying to me that, after more than 30 years since the first edition of *Introductory Combinatorics* was published, it continues to be well received by many people in the professional mathematical community.

I am very grateful to many individuals who have given me comments on previous editions and for this edition, including the discovery of typos. These individuals include, in no particular order: Russ Rowlett, James Sellers, Michael Buchner, Leroy F. Meyers, Tom Zaslavsky, Nils Andersen, James Propp, Louis Deaett, Joel Brawley. Walter Morris, John B. Little, Manley Perkel, Cristina Ballantine, Zixia Song, Luke Piefer, Stephen Hartke, Evan VanderZee, Travis McBride, Ben Brookins, Doug Shaw, Graham Denham, Sharad Chandarana, William McGovern, and Alexander Zakharin. Those who were asked by the publisher to review the fourth edition in preparation for this fifth edition include Christopher P. Grant who made many excellent comments. Chris Jeuell sent me many comments on the nearly completed fifth edition and saved me from additional typos. Mitch Keller was an excellent accuracy checker. Typos, but I hope no mistakes, probably remain and they are my responsibility. I am grateful to everyone who brings them to my attention. Yvonne Nagel was extremely helpful in solving a difficult problem with fonts that was beyond my expertise.

It has been a pleasure to work with the editorial staff at Prentice Hall, namely, Bill Hoffman, Caroline Celano, and especially Raegan Heerema, in bringing this fifth edition to completion. Pat Daly was a wonderful copyeditor.

The book, I hope, continues to reflect my love of the subject of combinatorics, my enthusiasm for teaching it, and the way I teach it.

Finally, I want to thank again my dear wife, Mona, who continues to bring such happiness, spirit, and adventure into my life.

Richard A. Brualdi
Madison, Wisconsin

Contents

Chapter 1

What Is Combinatorics?

It would be surprising indeed if a reader of this book had never solved a combinatorial problem. Have you ever counted the number of games n teams would play if each team played every other team exactly once? Have you ever attempted to trace through a network without removing your pencil from the paper and without tracing any part of the network more than once? Have you ever counted the number of poker hands that are full houses in order to determine the odds against a full house? More recently, have you ever solved a Sudoku puzzle? These are all combinatorial problems. As these examples might suggest, combinatorics has its roots in mathematical recreations and games. Many problems that were studied in the past, either for amusement or for their aesthetic appeal, are today of great importance in pure and applied science. Today, combinatorics is an important branch of mathematics. One of the reasons for the tremendous growth of combinatorics has been the major impact that computers have had and continue to have in our society. Because of their increasing speed, computers have been able to solve large-scale problems that previously would not have been possible. But computers do not function independently. They need to be programmed to perform. The bases for these programs often are combinatorial algorithms for the solutions of problems. Analysis of these algorithms for efficiency with regard to running time and storage requirements demands more combinatorial thinking.

Another reason for the continued growth of combinatorics is its applicability to disciplines that previously had little serious contact with mathematics. Thus, we find that the ideas and techniques of combinatorics are being used not only in the traditional area of mathematical application, namely the physical sciences, but also in the social sciences, the biological sciences, information theory, and so on. In addition, combinatorics and combinatorial thinking have become more and more important in many mathematical disciplines.

Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules. Two general types of problems occur repeatedly:

- *Existence of the arrangement.* If one wants to arrange the objects of a set so that certain conditions are fulfilled, it may not be at all obvious whether such an arrangement is possible. This is the most basic of questions. If the arrangement is not always possible, it is then appropriate to ask under what conditions, both necessary and sufficient, the desired arrangement can be achieved.

- *Enumeration or classification of the arrangements.* If a specified arrangement is possible, there may be several ways of achieving it. If so, one may want to count or to classify them into types.

If the number of arrangements for a particular problem is small, the arrangements can be listed. It is important to understand the distinction between listing all the arrangements and determining their number. Once the arrangements are listed, they can be counted by setting up a one-to-one correspondence between them and the set of integers $\{1, 2, 3, \ldots, n\}$ for some n. This is the way we count: one, two, three, ... However, we shall be concerned primarily with techniques for determining the number of arrangements of a particular type without first listing them. Of course the number of arrangements may be so large as to preclude listing them all.

Two other combinatorial problems often occur.

- *Study of a known arrangement.* After one has done the (possibly difficult) work of constructing an arrangement satisfying certain specified conditions, its properties and structure can then be investigated.

- *Construction of an optimal arrangement.* If more than one arrangement is possible, one may want to determine an arrangement that satisfies some optimality criterion—that is, to find a "best" or "optimal" arrangement in some prescribed sense.

Thus, a general description of combinatorics might be that *combinatorics is concerned with the existence, enumeration, analysis, and optimization of discrete structures.* In this book, *discrete* generally means "finite," although some discrete structures are infinite.

One of the principal tools of combinatorics for verifying discoveries is *mathematical induction.* Induction is a powerful procedure, and it is especially so in combinatorics. It is often easier to prove a stronger result than a weaker result with mathematical induction. Although it is necessary to verify more in the inductive step, the inductive hypothesis is stronger. Part of the art of mathematical induction is to find the right *balance of hypotheses and conclusions* to carry out the induction. We assume that the reader is familiar with induction; he or she will become more so as a result of working through this book.

The solutions of combinatorial problems can often be obtained using ad hoc arguments, possibly coupled with use of general theory. One cannot always fall back on application of formulas or known results. A typical solution of a combinatorial problem might encompass the following steps: (1) Set up a mathematical model, (2) study the model, (3) do some computation for small cases in order to develop some confidence and insight, and (4) use careful reasoning and ingenuity to finally obtain the solution of the problem. For counting problems, the inclusion–exclusion principle, the so-called pigeonhole principle, the methods of recurrence relations and generating functions, Burnside's theorem, and Pólya's counting formula are all examples of general principles and methods that we will consider in later chapters. Often, however, cleverness is required to see that a particular method or formula can be applied and how to apply. Thus, experience in solving combinatorial problems is very important. *The implication is that with combinatorics, as with mathematics in general, the more problems one solves, the more likely one is able to solve the next problem.*

We now consider a few introductory examples of combinatorial problems. They vary from relatively simple problems (but whose solution requires ingenuity) to problems whose solutions were a major achievement in combinatorics. Some of these problems will be considered in more detail in subsequent chapters.

1.1 Example: Perfect Covers of Chessboards

Consider an ordinary chessboard which is divided into 64 squares in 8 rows and 8 columns. Suppose there is available a supply of identically shaped dominoes, pieces which cover exactly two adjacent squares of the chessboard. Is it possible to arrange 32 dominoes on the chessboard so that no 2 dominoes overlap, every domino covers 2 squares, and all the squares of the chessboard are covered? We call such an arrangement a *perfect cover* or *tiling* of the chessboard by dominoes. This is an easy arrangement problem, and we can quickly construct many different perfect covers. It is difficult, but nonetheless possible, to count the number of different perfect covers. This number was found by Fischer[1] in 1961 to be $12,988,816 = 2^4 \times 17^2 \times 53^2$. The ordinary chessboard can be replaced by a more general chessboard divided into mn squares lying in m rows and n columns. A perfect cover need not exist now. Indeed, there is no perfect cover for the 3-by-3 board. For which values of m and n does the m-by-n chessboard have a perfect cover? It is not difficult to see that an m-by-n chessboard will have a perfect cover if and only if at least one of m and n is even or, equivalently, if and only if the number of squares of the chessboard is even. Fischer has derived general formulas involving trigonometric functions for the number of different perfect covers for the m-by-n chessboard. This problem is equivalent to a famous problem in molecular physics known as the *dimer problem.* It originated in the investigation of the absorption of diatomic atoms (dimers) on surfaces. The squares of the chessboard correspond to molecules, while the dominoes correspond to the dimers.

[1]M. E. Fischer, Statistical Mechanics of Dimers on a Plane Lattice, *Physical Review*, 124 (1961), 1664–1672.

Consider once again the 8-by-8 chessboard and, with a pair of scissors, cut out two diagonally opposite corner squares, leaving a total of 62 squares. Is it possible to arrange 31 dominoes to obtain a perfect cover of this "pruned" board? Although the pruned board is very close to being the 8-by-8 chessboard, which has over 12 million perfect covers, it has no perfect cover. The proof of this is an example of simple, but clever, combinatorial reasoning. In an ordinary 8-by-8 chessboard, usually the squares are alternately colored black and white, with 32 of the squares colored white and 32 of the squares colored black. If we cut out two diagonally opposite corner squares, we have removed two squares of the same color, say white. This leaves 32 black and 30 white squares. But each domino will cover one black and one white square, so that 31 nonoverlapping dominoes on the board cover 31 black and 31 white squares. We conclude that the pruned board has no perfect cover. The foregoing reasoning can be summarized by

$$31\boxed{\mathrm{B}\,|\,\mathrm{W}} \neq 32\boxed{\mathrm{B}} + 30\boxed{\mathrm{W}}.$$

More generally, we can take an m-by-n chessboard whose squares are alternately colored black and white and arbitrarily cut out some squares, leaving a pruned board of some type or other. When does a pruned board have a perfect cover? For a perfect cover to exist, the pruned board must have an equal number of black and white squares. But this is not sufficient, as the example in Figure 1.1 indicates.

W	×	W	B	W
×	W	B	×	B
W	B	×	B	W
B	W	B	W	B

Figure 1.1

Thus, we ask: What are necessary and sufficient conditions for a pruned board to have a perfect cover? We will return to this problem in Chapter 9 and will obtain a complete solution. There, a practical formulation of this problem is given in terms of assigning applicants to jobs for which they qualify.

There is another way to generalize the problem of a perfect cover of an m-by-n board by dominoes. Let b be a positive integer. In place of dominoes we now consider 1-by-b pieces that consist of b 1-by-1 squares joined side by side in a consecutive manner. These pieces are called *b-ominoes.* and they can cover b consecutive squares in a row or b consecutive squares in a column. In Figure 1.2, a 5-omino is illustrated. A 2-omino is simply a domino. A 1-omino is also called a *monomino.*

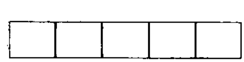

Figure 1.2 A 5-omino