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ROBUST CONTROL DESIGN 2000

A Proceedings volume from the 3rd IFAC Symposium
Prague, Czech Republic, 21 - 23 June 2000

Edited by

V. KUČERA and M. ŠEBEK

Volume 2

PERGAMON

ROBUST CONTROL DESIGN 2000
(ROCOND 2000)

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Foreword

The 3rd IFAC ROCOND Symposium on Robust Control Design took place in Hotel Renaissance, Prague, Czech Republic during June 21 – 23, 2000.

The ROCOND symposia have developed from an IFAC conference on Control System Design, which was held in Zűrich, Switzerland in 1991. The 1st IFAC ROCOND Symposium was organized in Rio de Janeiro, Brazil in 1994. The series then continued by the 2nd IFAC ROCOND Symposium in Budapest, Hungary in 1997.

Following the tradition, the aim of ROCOND 2000 was to bring together the robust control community to discuss the trends in the field and to present new methods and applications.

The technical program included 21 sessions on robust control and related topics in identification and signal processing. The methods presented in these sessions included linear matrix inequalities, polynomial techniques, sliding modes, optimal control, fuzzy and adaptive control. Attention was paid to linear as well as nonlinear systems.

The highlights of the technical program were two plenary lectures by world famous experts in the field: Robust Control and Filtering Design for Discrete-Time Systems, by J. Geromel (Universidade Estadual de Campinas, Brazil) and H2-optimization: Theory and Application to Robust Control Design, by H. Kwakernaak (Twente University, Netherlands).

Very well attended was the invited application session on Parameter-Space Tools for Robust Control, organized by J. Ackermann (German Aerospace Research Establishment, Oberpfaffenhofen). Part of the Symposium was an Europoly Workshop, which included 3 sessions of high quality papers on the theory and applications of polynomial design methods in control and signal processing.

It is my pleasure to congratulate U. Shaked (Tel Aviv University, Israel), the chairman of the International Program Committee and M. Šebek (Czech Technical University in Prague, Czech Republic), the chairman of the National Organizing Committee for their achievements in preparing and running the Symposium. I hope that the robust control community enjoyed this event and already look forward to the 4th IFAC ROCOND Symposium to be held in Milano, Italy in the year 2003.

Vladimír Kučera
Editor

CONTENTS

VOLUME 1

POLYNOMIAL METHODS, SINGULAR PERTURBATIONS

FILTERING

POSTER PAPERS I

LMI TECHNIQUES

SLIDING MODES

POSTER PAPERS II

SYSTEMS WITH DELAY

IDENTIFICATION

ROBUST CONTROL I

ROBUST CONTROL II

ROBUST CONTROL III

VOLUME 2

NONLINEAR SYSTEMS

PLENARY PAPER

H_∞ AND H_2 CONTROL

APPLICATION I

PARAMETER-SPACE TOOLS FOR ROBUST CONTROL

APPLICATION II

FUZZY, QFT AND ADAPTIVE CONTROL

APPLICATION III

EUROPOLY I

EUROPOLY II

EUROPOLY III

BI-QUADRATIC STABILITY FOR NONLINEAR SYSTEMS *

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Abstract: This paper deals with the robust stability of nonlinear systems with real time varying parameters having both magnitude and rate of variation which are confined to a given polytope. The system matrices may have entries which are rational functions of the states and uncertain parameters. LMI conditions are given, when they are feasible, they garantee the asymtotic stability of the origin of the system through a Lyapunov function of the type $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ where the matrix function $\mathcal{P}(x,\delta)$ depends quadratically on the states (x) and uncertain parameters (δ). The problem of maximizing an estimate of the region of attraction is also presented and a numerical example is used to show the potential of the proposed results. *Copyright ©2000 IFAC*

1. INTRODUCTION

Due to its nice properties, the LMI framework has been largely used to solve many robust control and filtering problems. In the context of linear uncertain systems many important results are now available (Boyd et al 1994). However, much work has to be done in order to extend these results to nonlinear systems.

In this last few years some interesting LMI based results for nonlinear systems have appeared in the literature. See for instance (Blanchini 1999),(El Ghaoui and Scorletti 1995).

For nonlinear systems, the following two aspects may contribute to the conservativeness of the existing LMI based results.

The first one is that LMIs, for nonlinear systems, are state dependent, and conditions of the type $x'\mathcal{P}(x,\delta)x > 0$ cannot be directly replaced by the state & parameter dependent LMI $\mathcal{P}(x,\delta) > 0$ without a great deal of conservatism.

Another potential source of conservatism is the use of quadratic Lyapunov functions for nonlinear systems. In general, quadratic Lyapunov functions are suited to show the stability in a small neighbourhood of the origin. If the size of the region of attraction is an important issue, more complex Lyapunov functions could lead to larger estimates of the region of attraction. Of course, it cames at the expense of a more intensive computation.

In this paper we present results to reduce these two potential conservative aspects listed above. It is shown how to reduce the conservativeness of state dependent LMIs and, instead of quadratic Lyapunov functions, it is utilized a more general function of the type $x'\mathcal{P}(x,\delta)x$ where $\mathcal{P}(x,\delta)$ is a quadratic matrix function of (x,δ).

The problem of concern consists of analysing the local stability with respect to a given polytopic neigbourhood of the origin under the assumption that the uncertain parameters and their respective rate of variation, are bounded by a given polytope that represents their admissible values. The problem of enlarging an estimate of the region of attraction inside the polytopic neigbourhood is also addressed and LMI solutions are proposed. The class of systems considered in this paper is described by a model in which the system matrices are allowed to have entries which are rational functions of the states and uncertain parameters. This is a large class of systems that includes

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the rational systems treated in (El Ghaoui and Scorletti 1995) and (Sasaki and Uchida 1997).
The paper is organized as follows. The next section is devoted to some preliminaries. Some definitions are presented and Lemma 2.1 shows how to reduce the conservativeness of state dependent LMIs. The problem of concern and the class of systems we deal with are presented in section 3. The stability results are stated in section 4 based on a notion called bi-quadratic stability. In section 5 we present the results for enlarging an estimate of the region of attraction and an example is presented to illustrate the results. Some concluding remarks end the paper. Due to space limitation, some results, examples and remarks and most part of the references were removed. We reffer the reader to the full version of this paper for more details.

2. PRELIMINARIES

Let us start with some definitions.
Consider the system

$$\dot{x} = A(\delta, x)\, x \tag{1}$$

where $A(\delta, x)$ denotes a matrix function of x, δ which are the state and uncertain parameter vectors respectively. Let $\mathcal{B}_x$, $\mathcal{B}_\delta$ be given polytopes representing, respectively, a neighbourhood of the origin and the admissible values of the uncertain parameters. It is assumed that the right handside of (1) is bounded $\forall x \in \mathcal{B}_x,\ \delta \in \mathcal{B}_\delta$. Depending on the context, the polytope $\mathcal{B}_\delta$ will represent the admissible values of δ and its variation rate $\dot{\delta}$. In this case we use the notation $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$. The notation $(x, \delta, \dot{\delta}) \in \mathcal{B}$ means that $x \in \mathcal{B}_x$ and $(\delta, \dot{\delta}) \in \mathcal{B}_\delta$.
The notation used in this paper is standard. I_r and 0_r denote the $r \times r$ identity and zero matrices. The matrices dimensions will not be specified when the dimensions can be deduced from the context. $A > 0$ (≥ 0) means that A is a symmetric positive definite (semi-definite) matrix. The time derivative of a function $R(t)$ will be denoted by $\dot{R}(t)$ and the argument (t) will be always ommited.

The problem of concern in this paper is to analyse the local stability of the equilibrium point $x = 0$ system (1). The stability notion that will be used is reffered to as Bi-quadratic stability.

Definition 2.1. (Bi-Quadratic stability). Let $\mathcal{B}_x, \mathcal{B}_\delta$ be given polytopes.
The origin of the system (1) is said to be ***locally bi-quadratically stable*** if there exist class $\mathcal{K}$ functions $\phi_1(.), \phi_2(.)$ and a function of the type $v(x, \delta) = x'\mathcal{P}(x, \delta)x$, where $\mathcal{P}(x, \delta)$ is a quadratic matrix function of (x, δ), such that the following conditions are satisfied $\forall\ (\delta, \dot{\delta}) \in \mathcal{B}_\delta,\ ,\forall x \in \mathcal{B}_x$:

- $v(x, \delta) = x'\mathcal{P}(x, \delta)x \geq \phi_1(\|x\|)$
- $\dot{v}(x, \delta) = x'\dot{\mathcal{P}}(x, \delta)\,x + x'A(x, \delta)'\,\mathcal{P}(x, \delta)x + x'\mathcal{P}(x, \delta)A(x, \delta)'\,x \leq -\phi_2(\|x\|)$

In the affirmative case, $v(x, \delta) = x'\mathcal{P}(x, \delta)x$ is said to be a local Lyapunov function for the origin.

△△△

Note that Bi-Quadratic stability implies asymptotic stability of the origin and it may be viewed as an extension, to nonlinear systems, of a stability notion introduced in (Trofino and de Souza 1999) for the linear case. The interest of this definition is that the Lyapunov function employs a state&parameter dependent matrix instead of a fixed one used in the quadratic stability notion. Moreover, the usual quadratic stability definition can be recovered as a special case. As usual in nonlinear systems theory, once we have a Lyapunov function $v(x, \delta)$ we may get an estimate of the region of attraction of the equilibrium point. In section 5 we present techniques to solve the problem of maximizing the size of the region of attraction to be estimated.
Now we turn our attention to another problem that arises in the analysis of nonlinear system via LMI methods.
As previously mensioned, the resulting LMIs for nonlinear systems are state dependent and this may lead the standard techniques develloped for linear systems to fail if they are directly applied to nonlinear systems. For instance, the condition $x'\mathcal{P}(x, \delta)x > 0$, may be tested with the powerfull algorithms developed for LMIs if $\mathcal{P}(x, \delta)$ is affine in (x, δ). In this case we just need to test whether the LMI condition $\mathcal{P}(x, \delta) > 0$ is satisfied or not $\forall\ \delta \in \mathcal{B}_\delta, x \in \mathcal{B}_x$, i.e. at the vertices of the meta polytope $\mathcal{B} = \mathcal{B}_\delta \times \mathcal{B}_x$. However, due to the LMI state dependence, this last LMI condition is, in general, too conservative to be used in lieu of $x'\mathcal{P}(x, \delta)x > 0$. To illustrate this fact, let us consider the nonlinear function $g(x) = x'H(x)x$ where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}'$ and

$$H(x) = \begin{bmatrix} 1 + x_2 & -\dfrac{x_1 + x_2}{2} \\ -\dfrac{x_1 + x_2}{2} & 1 + x_1 \end{bmatrix} \tag{2}$$

It is simple to verify that $g(x) = x_1^2 + x_2^2$ and thus $g(x) > 0\ \forall x \neq 0$. However, the condition $H(x) > 0\ \forall x \neq 0$ is not satisfied. The following lemma presents a less restrictive LMI test to check if the condition $x'H(x)x > 0$ holds $\forall x \in \mathcal{B}_x,\ x \neq 0$.

Lemma 2.1. Let $H(z)$ be a matrix whose entries are affine functions of $z = \begin{bmatrix} z_1 & \ldots & z_{n_z} \end{bmatrix}' \in \Re^{n_z}$ and $\mathcal{B}_z$ be a given convex polytope. Consider the nonlinear scalar function $z'H(z)z$ and the auxiliary matrix function $C_z\ :\ \Re^{n_z} \mapsto \Re^{(n_z-1)\times n_z}$ indicated below.

$$C_z = \begin{bmatrix} z_2 & -z_1 & 0 & 0 & \dots & 0 \\ 0 & z_3 & -z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & z_{n_z} & -z_{n_z-1} \end{bmatrix} \quad (3)$$

Then the condition $z'H(z)z > 0,\ \forall z \in \mathcal{B}_z,\ z \neq 0$ is satisfied if there exists a matrix L, with the same dimension of C_z', such that the following LMI condition is satisfied

$$H(z) + LC_z + C_z'L' > 0 \ , \quad \forall z \in \mathcal{B}_z \quad (4)$$

$\triangle\triangle\triangle$

The proof is simple and may be viewed as an application of the Finsler lemma (Boyd et al 1994) if we notice that, by construction, we have $C_z\, z = 0, \forall z$. A formal proof may be found in (Trofino 2000).

To illustrate the interest of lemma 2.1, let us return to the nonlinear function $g(x) = x'H(x)x$ with $H(x)$ given by (2). It was shown that despite $g(x) > 0\ \forall x \neq 0$, the condition $H(x) > 0\ \forall x \neq 0$ is not satisfied. Using the Lemma 2.1 with $C_x = \begin{bmatrix} x_2 & -x_1 \end{bmatrix}$ and the following choice for $L = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}'$ we may show that indeed $g(x) > 0\ \forall x \neq 0$ because $H(x) + LC_x + C_x'L' = I_2 > 0\ \forall x$. Although the conditions of the lemma 2.1 are only sufficient, for this particular example they are not conservative.

3. SYSTEM REPRESENTATION

Let us consider that the system (1) may be rewritten in a equivalent form as indicated below.

$$\begin{cases} \dot{x} = A(\delta, x)\, x = \sum_{i=0}^{q} A_i\, \pi_i \ , \quad \pi_0 = x \\ \pi_{i+1} = \Theta_i(\delta, x)\, \pi_i \ , \quad i = 0, \dots, r-1 \\ \Omega(\delta, \pi)\, \pi = 0 \end{cases} \quad (5)$$

where $x \in \Re^{m_0}$ denotes the state vector; $\delta \in \Re^{n_\delta}$ the vector of uncertain parameters; $\pi_i \in \Re^{m_i}$, $\pi = \begin{bmatrix} \pi_0' & \dots & \pi_q' \end{bmatrix}' \in \Re^m$ are auxiliary functions of (x, δ) representing the nonlinearities of the system; $\Theta_i(\delta, x) \in \Re^{m_{i+1} \times m_i}$, for $i = 0, \dots, r-1$ $(r \leq q)$ are affine function matrices of (δ, x) used to express the relations between the vectors π_i; $A_i \in \Re^{m_0 \times m_i}$, for $i = 0, \dots, q$ are fixed matrices of structure, and $\Omega(\delta, \pi) \in \Re^{m_\Omega \times m}$ is an affine function matrix of (δ, π) used to express additional relations between the vectors π_i.

To simplify the notation we always use π_i, Θ_i, Ω without expliciting their respective dependence on x, δ, and time.

Hereafter we assume that the right handside of the diffential equation in (5) is bounded for all values of π of interest and that the system representation in terms of the auxiliary variable π is equivalent to the representation $\dot{x} = A(\delta, x)\, x$. In other words we are assuming that the auxiliary variable π may be eliminated from the expressions and if this is done we recover the original system representation $\dot{x} = A(\delta, x)\, x$.

The system representation (5) may be used to represent a large class of uncertain nonlinear systems of the type $\dot{x} = A(\delta, x)\, x$.

For instance, the class of bi-linear system where $A(x) = \sum_{i=1}^{m_0} \tilde{A}_i\, x_i$, where $x_i \in \Re$ are the entries of x and $\tilde{A}_i$ are fixed matrices , may be written in the form (5) with the choice $q = 1$, $A_1 = \begin{bmatrix} \tilde{A}_1 & \dots & \tilde{A}_{m_0} \end{bmatrix}$, $\Theta_0(x) = \begin{bmatrix} x_1 I_{m_0} & \dots & x_{m_0} I_{m_0} \end{bmatrix}'$.

Notice that, in general, the system representation in terms of the π variables is not unique. The dimension of the π vector plays an important role in the results of the next sections. In general, as the dimension of the π vector increases, the conservativeness decreases because more decision variables to be tuned are introduced in the underlying problem. However, this came at the expense of a more intensive computation.

The case where $A(x, \delta)$ is a polynomial function of (x, δ) may be also represented as in (5). This is shown in the example 5.1. In the following we consider the case where $A(\delta, x)$ is a rational function of (x, δ).

Example 3.1. A simple fermentation process in a batch reactor may be described by the equations (Bastin and Douchain 1990)

$$\dot{x}_b = \mu x_b,\ \dot{s} = -k_1 \mu x_b,\ \mu = \frac{\mu_0 s}{k_m + s + \frac{s^2}{k_1}} \quad (6)$$

where the states are the biomass concentration x_b and the susbstrate concentration s; μ is the population growth rate and k_1, k_m, μ_0 are constants of the fermentation process. Although we may consider these constants as uncertainties, for simplicity in the presentation we assume these constants are known.

In order to represent the system (6) as in (5) we may define $r = 0$ and

$$\Theta_0 = \begin{bmatrix} 0 & s \end{bmatrix} \ , \ x = \pi_0 = \begin{bmatrix} x_b \\ s \end{bmatrix} \ , \ \pi_1 = \Theta_0 \pi_0$$

$$\pi_2 = \mu \pi_0, \quad \pi' = \begin{bmatrix} x' & \pi_1 & \pi_2' & \mu \end{bmatrix}$$

From the expression of μ we get $\mu k_m + \mu s + \mu \frac{s^2}{k_1} - \mu_0\, s = 0$ which in turn yields the representation (5) as follows:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ -k_1 & 0 \end{bmatrix} \pi_2 \ , \quad \pi_1 = \Theta_0 \pi_0$$

$$\Omega = \begin{bmatrix} \mu I_2 & 0 & -I_2 & 0 \\ [0 \ \ -\mu_0] & \mu / k_1 & [0 \ \ 1] & k_m \end{bmatrix}$$

4. BI-QUADRATIC STABILITY RESULTS

In order to present the results, some considerations and some auxiliary notations are needed. Notice from (5) that Θ_i are affine function matrices of (x,δ). In particular, the matrix Θ_0 plays an important role in bi-quadratic stability methods since the Lyapunov function will be Θ_0 dependent, as can be seen from definition 2.1. Let us represent Θ_0 as follows:

$$\Theta_0 = \sum_{i=1}^{m_0} T_i x_i + \sum_{i=1}^{n_\delta} U_i \delta_i + V \qquad (7)$$

where T_i, U_i, V are constant matrices of structure having the same dimensions of Θ_0 and x_i, δ_i are the entries of the vectors x and δ respectively.

Theorem 4.1. Consider the uncertain nonlinear system (5), the matrix C_x obtained from the definition (3), the representation of Θ_0 from (7) , let E_i be the i-th row of I_{m_0} and the auxiliary notation

$$\tilde{\Theta}_0 = \sum_{i=1}^{m_0} T_i\, x\, E_i \quad , \quad \hat{\Theta}_0 = \sum_{i=1}^{n_\delta} U_i \dot{\delta}_i$$

$$A_{a_1} = \begin{bmatrix} A_0 & A_1 \\ \hat{\Theta}_0 + (\Theta_0 + \tilde{\Theta}_0)A_0 & (\Theta_0 + \tilde{\Theta}_0)A_1 \end{bmatrix} \qquad (8)$$

$$A_{a_2} = \begin{bmatrix} A_2 & \dots & A_q \\ (\Theta_0 + \tilde{\Theta}_0)A_2 & \dots & (\Theta_0 + \tilde{\Theta}_0)A_q \end{bmatrix}$$

$$C_{a_1} = \begin{bmatrix} C_x & 0 \\ \Theta_0 & -I_{m_1} \end{bmatrix}$$

$$C_{a_2} = \begin{bmatrix} & & \Omega & & \\ C_x & 0 & 0 & \dots & 0 \\ \Theta_0 & -I_{m_1} & 0 & \dots & 0 \\ 0 & \Theta_1 & -I_{m_2} & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & \Theta_{r-2} & -I_{m_{r-1}} & 0 \\ 0 & \dots & 0 & \Theta_{r-1} & -I_{m_r} \end{bmatrix}$$

Let $\mathcal{B}$ be a given convex polytope and suppose that $(\pi, \delta, \dot{\delta}) \in \mathcal{B}$. Then the system (5) is locally Bi-Quadratically stable if there exist matrices P, L_1, L_2, with the same dimensions of $A_{a_1}, C'_{a_1}, C'_{a_2}$ respectively, such that the following LMIs are satisfied for $(\pi, \delta, \dot{\delta})$ at the vertices of the polytope $\mathcal{B}$.

$$\begin{bmatrix} A'_{a_1}P + PA_{a_1} & PA_{a_2} \\ A'_{a_2}P & 0 \end{bmatrix} + L_2 C_{a_2} + C'_{a_2} L'_2 < 0 \qquad (9)$$

$$P + L_1 C_{a_1} + C'_{a_1} L'_1 > 0$$

In the affirmative case, the function $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ with $\mathcal{P}(x,\delta)$ given below is a local Lyapunov function for the system (5).

$$\mathcal{P}(x,\delta) = \begin{bmatrix} I_{m_0} \\ \Theta_0(x,\delta) \end{bmatrix}' P \begin{bmatrix} I_{m_0} \\ \Theta_0(x,\delta) \end{bmatrix} \qquad (10)$$

△△△

Proof: Consider the system (5) and define the following partition of the π vector:

$$\xi_1 = \begin{bmatrix} x \\ \pi_1 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} \pi_2 \\ \vdots \\ \pi_q \end{bmatrix}, \ \pi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad (11)$$

Notice that, by construction, we have $C_x x = 0$. Moreover, the constraint $C_{a_1}\xi_1 = 0$ is equivalent to $\pi_1 = \Theta_0 \pi_0$ and $C_x x = 0$. Similarly, the constraint $C_{a_2}\pi = 0$ is equivalent to $\pi_{i+1} = \Theta_i \pi_i$, $C_x x = 0$ and $\Omega\pi = 0$. Except for the equality constraint $C_x x = 0$, which is always satisfied by construction, the remaining equalities are used to define the system representation in (5).

Suppose now the conditions of the Theorem 4.1 are satisfied. Then there exist a sufficient small positive scalar ϵ_1 such that $P + L_1 C_{a_1} + C'_{a_1} L'_1 - \epsilon_1 I > 0$ still holds. In a similar way it is possible to add a term $\epsilon_2 I$ to the first LMI in (9). Now take the first LMI of (9) modified with the added term $\epsilon_2 I$. Pre and post multiply it by π' and its transpose respectively. Subsequently pre and post multiply $P + L_1 C_{a_1} + C'_{a_1} L'_1 - \epsilon_1 I > 0$ by ξ'_1 and its transpose respectively. These two operations lead the following to be satisfied.

$$\xi'_1 P \xi_1 - \epsilon_1 \xi'_1 \xi_1 > 0, \ \forall (\pi, \delta, \dot{\delta}) \in \mathcal{B} : \ C_{a_1}\xi_1 = 0$$

$$\xi'_1(A'_{a_1}P + PA_{a_1})\xi_1 + \xi'_1 PA_{a_2}\xi_2 + \xi'_2 A'_{a_2} P \xi_1 + \epsilon_2 \pi'\pi < 0 \ , \ \forall (\pi, \delta, \dot{\delta}) \in \mathcal{B} : \ C_{a_2} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \qquad (12)$$

Since $\xi'_1 = \begin{bmatrix} x' & \pi'_1 \end{bmatrix} = \begin{bmatrix} x' & x'\Theta'_0 \end{bmatrix}$ we get $||\xi_1|| \geq ||x||$ and thus $v(x,\Theta_0) = \xi'_1 P \xi_1 = x'\mathcal{P}(x,\delta)x > \epsilon_1 ||x||^2$, $\forall (\pi, \delta, \dot{\delta}) \in \mathcal{B}$, $x \neq 0$, where $\mathcal{P}(x,\delta)$ is the state&parameter dependent matrix indicated in(10). Note in addition that $||\pi|| \geq ||x||$.

Now we use the equalities

$$\xi'_1 PA_{a_1}\xi_1 = x'\mathcal{P}(x,\delta) \begin{bmatrix} A_0 & A_1 \end{bmatrix} \xi_1 + + x' \begin{bmatrix} I_{m_0} \\ \Theta_0 \end{bmatrix}' P \begin{bmatrix} 0 \\ \dot{\Theta}_0 \end{bmatrix} x$$

$$\xi'_1 PA_{a_2}\xi_2 = x'\mathcal{P}(x,\delta) \begin{bmatrix} A_2 & \dots & A_q \end{bmatrix} \xi_2$$

to show that the following holds.

$$\xi'_1(A'_{a_1}P + PA_{a_1})\xi_1 + \xi'_1 PA_{a_2}\xi_2 + \xi'_2 A'_{a_2} P \xi_1 = x'\dot{\mathcal{P}}(x,\delta)x + x'\mathcal{P}(x,\delta)(\sum_{i=0}^{q} A_i \pi_i) + (\sum_{i=0}^{q} A_i \pi_i)'\mathcal{P}(x,\delta)x$$

The above results together with (12) show that $\dot{v}(x,\delta) < -\epsilon_2 ||x||^2$. Thus $v(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function for the system (5) and therefore its origin is locally bi-quadratically stable.

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