Trends in Abstract and Applied Analysis
Volume 2

Nonlinear Interpolation and Boundary Value Problems

Paul W Eloe • Johnny Henderson

World Scientific

Trends in Abstract and Applied Analysis
Volume 2

Nonlinear Interpolation and Boundary Value Problems

Paul W Eloe
University of Dayton, USA
Johnny Henderson
Baylor University, USA

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Names: Eloe, Paul W., author. | Henderson, Johnny, author.
Title: Nonlinear interpolation and boundary value problems / Paul W. Eloe (University of Dayton, USA), Johnny Henderson (Baylor University, USA).
Description: Singapore ; Hackensack, NJ : World Scientific, [2016] | 2016 | Series: Trends in abstract and applied analysis ; volume 2 | Includes bibliographical references and index.
Identifiers: LCCN 2015040530| ISBN 9789814733472 (alk. paper) | ISBN 9814733474 (alk. paper)
Subjects: LCSH: Boundary value problems. | Differential equations.
Classification: LCC QA379 .E435 2016 | DDC 515/.35--dc23
LC record available at http://lccn.loc.gov/2015040530

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

In-house Editors: V. Vishnu Mohan/Kwong Lai Fun

Typeset by Stallion Press
Email: enquiries@stallionpress.com

Printed by FuIsland Offset Printing (S) Pte Ltd Singapore

Nonlinear Interpolation and Boundary Value Problems

TRENDS IN ABSTRACT AND APPLIED ANALYSIS

Series Editor: John R. Graef
The University of Tennessee at Chattanooga, USA

Published

Vol. 1 Multiple Solutions of Boundary Value Problems:
A Variational Approach
by John R. Graef & Lingju Kong

Vol. 2 Nonlinear Interpolation and Boundary Value Problems
by Paul W. Eloe & Johnny Henderson

Dedicated to the memory of Mary JoAnn Kramer Eloe.

Dedicated to David Ray Strunk III and Jana Elisabeth Strunk, and to the memory of their sister, Kathryn Madora Strunk.

Preface

The questions most often addressed concerning dynamical systems (such as differential equations, finite difference equations, dynamic equations on time scales, and so on) involve existence of solutions and uniqueness of solutions. The topics dealt with in this book address those two questions for boundary value problems for dynamical systems, but for the greater part, in the stated opposite order. That is, given a dynamical system

$$Ly = f, \tag{1}$$

satisfying boundary conditions,

$$l_i y = c_1, \quad 1 \le i \le n, \tag{2}$$

where $L : C^{(n)}(I) \to C(I)$ is an nth order linear differential operator (or difference operator or dynamic differential operator, etc.), I is an interval of the reals, f is nonlinear in $y^{(i-1)}$, $1 \le i \le n$, $l_i : C^{(n-1)}(I) \to \mathbb{R}$ are continuous and bounded, $c_i \in \mathbb{R}$, $1 \le i \le n$, and $n \ge 2$, the questions dealt with are **(i)** "When are solutions of (1)–(2) unique, when solutions exist?"; and **(ii)** "When does uniqueness of solutions of (1)–(2) imply existence of solutions of (1)–(2)?" Questions of type (ii) are called "uniqueness implies existence" questions. If (1) is a nonlinear ordinary differential equation, the answers to questions of type (i) and type (ii) are quite technical. If (1) is the simple equation,

$$y^{(n)}(x) = 0, \quad x \in I,$$

then the questions of type (i) and type (ii) are precisely those questions that motivate the study of polynomial interpolation. In the case of polynomial interpolation, a linear problem, the questions of type (i), when are solutions unique when they exist, or type (2), uniqueness implies existence, are equivalent.

An interpolation problem is the problem of selecting a unique function from a family of functions that satisfies prescribed conditions. Hence, to consider boundary value problems for ordinary differential equations as interpolation problems is appropriate and Hartman [43] employed the phrase, interpolation problems for nonlinear differential equations. For the purposes of the title of this book, we adopted a phrase, **nonlinear interpolation**. Specifically, for this book, we mean the following: the family of functions is the set of solutions of the nonlinear equation, (1), and the prescribed conditions are those given by the linear boundary conditions, (2). Since type (i) questions and type (ii) questions are apparently independent questions for nonlinear ordinary differential equations, we intend that the one phrase, nonlinear interpolation, contains both questions.

In routes taken to establish "uniqueness implies existence" for solutions of (1)–(2), uniqueness of solutions of one type of boundary value problem for (1) may imply uniqueness of solutions for related types of boundary value problems for (1), which can be ultimately useful in obtaining existence of solutions of (1)–(2). These types of results are sometimes referred to as "uniqueness implies uniqueness" results. These questions for nonlinear differential equations have a long history, with some motivational papers originating with Hartman [41], Lasota and Luczynski [86, 87] and Lasota and Opial [88] in the 1950's and 1960's. Soon after those papers, the most prominent such results for the case of (1) nonlinear were established by P. Hartman and L. Jackson, as well as by L. Jackson's students, throughout the 1960's and 1970's.

For a period in excess of 30 years, this book's authors have occupied the center of research attention devoted to "uniqueness implies uniqueness" and "uniqueness implies existence" questions for solutions of the boundary value problem (1)–(2). This book pulls together much of their work, along with closely related work by other authors, as well as includes some of the classical results as historical motivation. (The book by Agarwal [3], the paper by Agarwal [4] and the survey paper by Mawhin [90] contain documentation of some of the important historical works.)

In Chapter 1, conjugate boundary value problems and right focal boundary value problems are introduced for when (1) is an nth order ordinary differential equation. One goal of the chapter will be directed toward "uniqueness implies uniqueness" results for both conjugate boundary value problems and right focal boundary value problems. Yet, before those questions are addressed, the chapter is focused on a Kamke Theorem, a historically significant "Compactness Condition" on solutions of (1), and a result concerning solutions of boundary value problems and their continuous dependence on the boundary conditions. Each of these plays a fundamental role in the "uniqueness implies uniqueness" results of the chapter, as well as throughout many of the other chapters of the book. Some of the "uniqueness implies uniqueness" results for the cases of $n = 3, 4$, are necessary and sufficient results.

Chapter 2 is devoted to the "uniqueness implies existence" question for the boundary value problems introduced in Chapter 1; that is, the conjugate boundary value problems and the right focal boundary value problems. Historically motivational results for the conjugate boundary value problems for second and third order (1) comprise the first two sections of Chapter 2, with generalizations in the third section. The last two sections of the chapter focus on right focal problems. Throughout the chapter, extensive use is made of the "Compactness Condition" and continuous dependence of solutions on boundary conditions in conjunction with shooting methods, along with induction based on the arrangement indexing pattern for each family of boundary value problems.

Nonlocal boundary value problems for (1) are considered in Chapter 3. The first and third sections of this chapter deal with uniqueness of solutions of certain types of nonlocal boundary value problems leading to uniqueness of solutions for other nonlocal boundary value problems. The arguments for these "uniqueness implies uniqueness" results take different paths. Then, for each of the paths taken, in the second and fourth sections, use is made of the respective preceding uniqueness results in establishing "uniqueness implies existence" for the nonlocal boundary value problems. Again, the methods for existence include continuous dependence of solutions on boundary conditions, the "Compactness Condition" and shooting adapted to the unique nature of nonlocal boundary value problems.

The results of Chapter 4 are concentrated on the uniqueness of solutions implying their existence for discrete boundary value problems for finite difference equations. The addressed problems are, in some sense, discrete analogues of the results from Chapter 2, in that, discrete conjugate boundary value problems and discrete right focal boundary value problems are the focus. Discrete Lidstone boundary value problems are also dealt with for fourth order difference equations. Uniqueness hypotheses for solutions are viewed in terms of Hartman's *generalized zeros* and *discrete Rolle's Theorem*. The methods in each context rely strongly on induction and shooting, yet the inductions depend on the spacing of the discrete boundary points.

Chapter 5 contains results on "uniqueness implies existence" for solutions of boundary value problems for Hilger's dynamic equations on time scales. Given a nonempty closed subset of the reals, boundary value problems are defined on the subset in terms of the *Hilger delta derivative* (a general differentiation concept that unifies ordinary differentiation and discrete differences, with resulting equations called *dynamic equations*). Then, for dynamic equations of orders $n = 2, 3, 4$, uniqueness of solutions assumptions are imposed in terms of Bohner and Eloe's *generalized zeros* and *Rolle's Theorem on a time scale.* Shooting methods are adapted to the boundary value problems of the chapter, with the topological nature of the boundary points (such as *right dense boundary points*, *left dense boundary points*, *isolated boundary points*, and so on) playing major roles in the details of the existence arguments.

Chapter 6 contains a few remarks about "uniqueness implies uniqueness" and "uniqueness implies existence" for other boundary value problems along with some citations to works involving sufficient conditions for uniqueness of solutions, when solutions exist.

When many years of research result in a book such as this, there are always those to thank, who in one way or another, led to its writing. Paul W. Eloe thanks his wife, Laura. He also thanks his co-author, Johnny Henderson, for first, his extensive contributions in the study of nonlinear interpolation and boundary value problems and second, for the opportunity to contribute to this project which in many ways is a tribute to (the late) Lloyd K. Jackson. Johnny Henderson thanks first his wife, Darlene, and then others to whom he owes his thanks include David L. Skoug, Allan C. Peterson, James R. Wall, John V. Baxley, Overtoun M. Jenda, Jerry A. Veeh, August J. Garver, (the late) Wyth S. Duke, (the late) George M. Randall, (the late) Carroll Bo Sanderlin, (the late) Edward N. Mosely, (the late) Louis J. Grimm, (the late) Willam N. Hudson, (the late) William G. Leavitt, and (the late) Lloyd K. Jackson. Of course, he also thanks his co-author, Paul W. Eloe, not only for his collaboration on this book, but also for his enthusiastic and enlightening thoughts and comments while tossing around the baseball.

The authors especially want to thank their editors at World Scientific Publishing: Trends in Abstract and Applied Analysis Series Editor John R. Graef, Executive Editor Rochelle Kronzek, and Desk Editors V. Vishnu Mohan and Lai Fun Kwong.

Paul W. Eloe
Johnny Henderson

Contents

Chapter 1

Uniqueness Implies Uniqueness

This chapter is devoted primarily to uniqueness of solutions for boundary value problems for an ordinary differential equation. In particular, focus will be on when uniqueness of solutions for one type of boundary value problem leads to uniqueness of solutions for other types of boundary value problems. In addition to interest in the results themselves, such results sometimes play a role in establishing the existence of solutions (the topic of some of the subsequent chapters of this book). For example, in the case of linear boundary value problems for linear ordinary differential equations, uniqueness of solutions is equivalent to their existence.

The "uniqueness implies uniqueness" results in this chapter will be concentrated on conjugate boundary value problems and right focal boundary value problems. Boundary value problems, such as nonlocal boundary value problems, and some others will be dealt with in other chapters.

1.1 Some preliminaries

Throughout this chapter, we will consider boundary value problems for an nth order ordinary differential equation,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad a < x < b. \tag{1.1.1}$$

And throughout, we will use the following hypotheses concerning the differential equation:

(A) $f(x, r_1, \ldots, r_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous.
(B) Solutions of initial value problems for (1.1.1) are unique and extend to (a, b).

Given $2 \leq k \leq n$, $m_1, \ldots, m_k \in \mathbb{N}$ such that $\sum_{j=1}^{k} m_j = n$, points $a < x_1 < \cdots < x_k < b$, and $y_{ij} \in \mathbb{R}$, $1 \leq i \leq m_j$, $1 \leq j \leq k$, a boundary value problem for (1.1.1) satisfying

$$y^{(i-1)}(x_j) = y_{ij},\ 1 \le i \le m_j,\ 1 \le j \le k, \tag{1.1.2}$$

will be called either a *k-point conjugate boundary value problem*, or an $(m_1, \ldots, m_k)$ *conjugate boundary value problem.*

Also, given $2 \le r \le n$, $m_1, \ldots, m_r \in \mathbb{N}$ such that $\sum_{j=1}^{r} m_j = n$, $s_0 := 0$, $s_k := \sum_{j=1}^{k} m_j$, $1 \le k \le r$, points $a < x_1 < \cdots < x_r < b$, and $y_{ik} \in \mathbb{R}$, $s_{k-1} \le i \le s_k - 1$, $1 \le k \le r$, a boundary value problem for (1.1.1) satisfying

$$y^{(i)}(x_k) = y_{ik},\ s_{k-1} \le i \le s_k - 1,\ 1 \le k \le r, \tag{1.1.3}$$

will be called either an *r-point right focal boundary value problem*, or an $(m_1, \ldots, m_r)$ *right focal boundary value problem.* (We remark that, historically, such conditions were sometimes called *right* $(m_1, \ldots, m_r)$ *focal point boundary conditions*, but we will not use that terminology.)

Some of the results of this chapter, as well as the subsequent chapters on existence, rely on a type of "compactness condition" on sequences of solutions of (1.1.1). In particular, it was conjectured for many years that hypotheses (A) and (B), along with a uniqueness condition on $(1, 1, \ldots, 1)$ conjugate boundary value problems for (1.1.1) on (a, b), implied the following "compactness condition" on solutions:

(CP) If $\{y_k(x)\}$ is a sequence of solutions of (1.1.1) such that for some $[c, d] \subset (a, b)$ and some $M > 0$, $|y_k(x)| \le M$ on $[c, d]$, for all $k \ge 1$, then there exists a subsequence $\{y_{k_j}(x)\}$ such that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), for each $0 \le i \le n - 1$.

The conjecture was based in part on an existing proof by Jackson and Schrader [82] for the case of $n = 3$. In 1985, Schrader [102] presented a paper, during a Sectional Meeting of the American Mathematical Society held at the University of Missouri-Columbia, in which he revealed that he and L. Jackson had established the validity of (CP) under the assumptions of (A) and (B) and the uniqueness condition:

(C) There exists at most one solution of each $(1, 1, \ldots, 1)$ conjugate boundary value problem for (1.1.1) on (a, b).

Jackson and Schrader never published their result, yet written notes taken by A. C. Peterson of Jackson's presentation of the result in a seminar were communicated to J. Henderson. Because of its fundamental importance for this book, we will present the proof of the compactness result in its form as presented by Jackson in that seminar. (We remark that Agarwal [4] did publish the Jackson and Schrader proof.)

The proof that Jackson and Schrader gave made major use of results from a paper by Agronsky, Bruckner, Laczkovich and Preiss [5]. We now state some of the definitions and results from [5].

Definition 1.1.1. [[5], Agronsky *et al.*, p. 660] *Let*

$$P_n := \{p(x) : \mathbb{R} \to \mathbb{R} \mid p(x) \text{ is a polynomial of degree at most } n\}.$$

Definition 1.1.2. [[5], Agronsky *et al.*, p. 660] *Given* $E \subseteq [c, d]$, x_0 *is a* bilateral accumulation point *of* E, *in case* x_0 *is an accumulation point of both* $E \cap [c, x_0]$ *and* $E \cap [x_0, d]$.

Definition 1.1.3. [[5], Agronsky *et al.*, p. 666] *A function* $g : I \to \mathbb{R}$, *with* I *an interval, is said to be* n-convex (n-concave) on I *in case, for any distinct* $x_0, \dots, x_n \in I$, $\sum_{i=0}^{n} \frac{g(x_i)}{w'(x_i)} \geq 0$ (≤ 0), *where* $w(x) = \prod_{i=0}^{n}(x - x_i)$, (*so* $w'(x_j) = \prod_{i=0, i\neq j}^{n}(x_j - x_i)$).

Theorem 1.1.1. [[5], Agronsky *et al.*, p. 666] *Suppose* $g \in C^{(n)}(I)$. *Then* g *is* n*-convex on* I *if, and only if,* $g^{(n)}(x) \geq 0$, *on* I.

Theorem 1.1.2. [[5], Agronsky *et al.*, p. 666] g *is* n*-convex (*n*-concave) on* I *if, and only if,* $g \in C^{(n-2)}(I)$ *and* $g^{(n-2)}$ *is convex (concave).*

Theorem 1.1.3. [[5], Agronsky *et al.*, Thm. 13] *Let* $g \in C[c, d]$ *and assume that, for each* $p \in P_n$, *the set,* $\{x : p(x) = g(x)\}$, *does not have a bilateral accumulation point in* (c, d). *Then, there exists a subinterval* $I \subseteq [c, d]$ *on which* g *is either* $(n + 1)$*-convex or* $(n + 1)$*-concave.*

One final fundamental result is given prior to stating and proving the Jackson and Schrader "compactness condition." This result is known as the Kamke convergence theorem for solutions of initial value problems; the reader is referred to [42, Theorem 3.2, p. 14] or [77].

Theorem 1.1.4. [Kamke] *Assume that in a sequence of differential equations*

$$y^{(n)} = f_k(x, y, y', \dots, y^{(n-1)}), \quad k = 1, 2, \dots, \tag{1.1.4}$$

the functions $f_k(x, r_1, r_2, \dots, r_n)$ *are continuous on* $I \times \mathbb{R}^n$, *where* I *is an interval of the reals, and assume*

$$\lim_{k\to\infty} f_k(x, r_1, r_2, \dots, r_n) = f_0(x, r_1, r_2, \dots, r_n)$$

uniformly on each compact subset of $I \times \mathbb{R}^n$. *Assume that* $\{x_k\}_{k=0}^{\infty} \subset I$ *converges to* x_0. *For each* $k = 1, 2, \ldots$, *let* y_k *denote a solution of* (1.1.4), *for the respective* k, *defined on a maximal interval* $I_k \subset I$ *where* $x_k \in I_k$. *Assume* $\lim_{k\to\infty} y_k^{(i-1)}(x_k) = y_i$ *for each* $i = 1, 2, \ldots, n$. *Then there is a subsequence* $\{y_{k_j}\}$ *of* $\{y_k\}$ *and there is a solution* y_0 *of*

$$y^{(n)} = f_0(x, y, y', \ldots, y^{(n-1)})$$

defined on a maximal interval $I_0 \subset I$ *such that* $x_0 \in I_0$, $y_0^{(i-1)}(x_0) = y_i$, $i = 1, 2, \ldots, n$, *and such that for any compact interval* $[c, d] \subset I_0$, *then* $[c, d] \subset I_{k_j}$ *eventually and* $\{y_{k_j}^{(i-1)}\}$ *converges uniformly to* $\{y_0^{(i-1)}\}$ *on* $[c, d]$, $i = 1, 2, \ldots, n$.

Now, we provide the Jackson and Schrader result known as the "compactness condition."

Theorem 1.1.5. [Jackson and Schrader] *If, with respect to* (1.1.1), *conditions* (A)–(C) *are satisfied, then condition* (CP) *is satisfied.*

Proof. Assume that $\{y_k(x)\}$ is a sequence of solutions of (1.1.1) which is uniformly bounded on some $[c, d] \subset (a, b)$. Then, by Helly's Selection (or Choice) Theorem [95, Helly's First Theorem, p. 222], there exists a subsequence $\{y_{k_j}(x)\}$ and a function $h \in BV[c, d]$ such that

$$\lim_{j\to\infty} y_{k_j}(x) = h(x) \text{ pointwise on } [c, d].$$

Set

$$H(x) := \int_c^x h(t)dt.$$

Then $H \in C[c, d]$, and by Theorem 1.1.3, either

(i) There exists a $p \in P_{n+1}$ such that $\{x : p(x) = H(x)\}$ has a bilateral accumulation point in (c, d),

or

(ii) H is $(n+2)$-convex or $(n+2)$-concave on some $[c_1, d_1] \subseteq [c, d]$.

We now relabel the subsequence $\{y_{k_j}(x)\}$ as $\{y_k(x)\}$.

And we first consider case (ii). It follows from Theorem 1.1.2 that $H \in C^{(n)}[c_1, d_1]$ and $H^{(n)}$ is convex, (or concave). Then, $H' \in C^{(n-1)}[c_1, d_1]$ and $H'(x) = h(x)$ *a.e.* on $[c_1, d_1]$. As a consequence of an application of the Schauder-Tychonoff fixed point theorem, $(1, \ldots, 1)$ conjugate boundary