# GENERALIZED FUNCTIONS, VOLUME 2

## SPACES OF FUNDAMENTAL AND GENERALIZED FUNCTIONS

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# Generalized Functions, Volume 2

## Spaces of Fundamental and Generalized Functions

# Preface to the Russian Edition

The constructions of Volume 1 proceeded on the basis of utilizing a few fundamental spaces: (1) the space $K$ of infinitely differentiable functions having compact supports; (2) the space $S$ of infinitely differentiable functions decreasing at infinity, together with all their derivatives, more rapidly than any power of $1/|x|$; (3) the space $Z$ of analytic functions $\varphi(z)$, satisfying inequalities of the form $z^k\varphi(z) \leqslant C_k e^{a|y|}$. Generalized functions—continuous linear functionals on these spaces—were adequate for the clarification of the fundamental features of the theory and for a number of simple, but important, applications to some questions of analysis, and in particular, to the theory of differential equations.

On the other hand, although we tried there to reduce to a minimum the number of spaces utilized, we did not succeed in bypassing one pair of spaces $K$ and $K'$: by considering generalized functions as continuous linear functionals in $K$, we inevitably had to consider their Fourier transforms as continuous linear functionals on $Z$. The advantages of such a viewpoint will be seen particularly clearly in Volume 5, where methods of complex variable function theory will render substantial assistance in algorithmic questions of the theory of generalized functions.

We shall need a considerably more extensive circle of spaces in Volume 3, which is devoted to deeper applications of the theory of generalized functions to differential equations, than those which we encountered periodically in Volume 1, and will meet here and there in Volume 2. Namely, applications of the theory of generalized functions to the Cauchy problem and to the problem of eigenfunction expansions will be elucidated in Volume 3. Here, the fundamental peculiarity of the theory of generalized functions, in that form in which we shall understand it in this book, will be completely apparent; it is that *different classes of problems require different classes of spaces*, and, indeed classes of spaces and not individual spaces.

Thus, uniqueness and existence theorems for the solution of the Cauchy problem for different partial differential equations require different spaces, which however possess some common properties. Problems of eigenfunction expansions for different differential operators also require different spaces which, nevertheless, have a number of common features. And similarly, boundary value problems for elliptic equations require their class of fundamental spaces and spaces of generalized functions.

In the preceding stage of development of functional analysis, which was connected with the theory of integral equations, the common base for the study of the various functional spaces encountered was the theory of linear normed spaces.*

Normed spaces turned out to be inadequate for the needs of the theory of generalized functions. It must not be thought that the situation is such that much more complex constructions would be required. It is directly opposite: among the normed spaces one does not find the simplest spaces, for example the spaces $K$ and $S$ possessing a whole series of essential properties.

In recent years the general theory of linear topological spaces has developed considerably. However, the most general linear topological spaces are rather complicated objects possessing a whole set of "pathological" properties, and are poorly adapted to the needs of the analyst.† The basis of the theory of generalized functions is the theory of the so-called *countably normed spaces (with compatible norms), their unions (inductive limits)*, and also of the *spaces conjugate to the countably normed ones or their unions*. This set of spaces is sufficiently broad on the one hand, and sufficiently convenient for the analyst on the other.

The theory of these spaces is expounded in Chapter I. Let us note that since the countably normed spaces are very close to normed spaces, a number of important theorems is obtained almost automatically by taking them over from the normed spaces into the countably normed spaces.‡ In reading this chapter it should be kept in mind that some of the theorems proved here are actually valid for more general spaces.

In the majority of questions the class of all countably normed spaces turns out to be too broad for the theory of generalized functions. Hence,

* However, even during this period works appeared which anticipated going beyond the limits of this class of spaces, the work of Köthe-Toeplitz and Köthe on spaces of sequences in the 30's, and also the work of Mazur and Orlicz.

† To the analyst it is natural to use estimates, not neighborhoods, which he inevitably reduces to some kind of estimates.

‡ Before reading this chapter it would be useful for the reader not acquainted with the theory of normed spaces to read the first three chapters, say, of the book "Elements of Functional analysis" by L. A. Lyusternik and V. I. Sobolev, Ungar, New York, 1961 or the first volume of the lectures "Elements of the Theory of Functions and Functional Analysis" by A. N. Kolmogorov and S. V. Fomin, Moscow University Press, Moscow, USSR, 1954.

in Chapter I we study the so-called *perfect* spaces (complete countably normed spaces in which the bounded sets are compact). The reader will meet a great number of examples of such spaces in the following chapters.

The reader will also find material referring to the general theory of countably normed spaces in the first three sections of Chapter IV in Volume 3.

The expounded viewpoint certainly excludes the possibility of an *a priori* description of all classes of spaces which may be encountered in connection with various problems of the theory of generalized functions: As we have already said above, each class of problems requires its own class of spaces. Therefore, essentially two classes of spaces are introduced and studied in Chapters II and IV: spaces of the type $K\{M_p\}$ in Chapter II; spaces of the type $S$ and similar spaces of type $W$ in Chapter IV. The spaces of type $S$ and $W$ essentially satisfy the demands of Chapters II and III of Volume 3 (the Cauchy problem), and spaces of type $K\{M_p\}$ the requirements of Chapter IV of Volume 3 (the problem of eigenfunction expansions). Chapter II, and, in part, Chapter III, of the present volume are devoted primarily to transferring the results of Chapters I and II of Volume 1, almost without any difficulty, to more general spaces. The spaces $K\{M_p\}$, which are natural illustrations of the general theory, appear here. On the other hand, the results of Chapter I permit the filling in of a whole series of essential gaps, in particular, the proof of the completeness of spaces of generalized functions on $K$, and the establishment of a number of new results, concerning for example the structure of generalized functions.

The theory of spaces of type $S$ is discussed in the last Chapter IV. These spaces which, as we have said already, are used in Volume 3 possess great internal orderliness, and we hope that even their independent study will give the analyst some satisfaction. The construction and utilization of these spaces is connected with results of the theory of quasi-analytic functions and the Phragmen-Lindelöf theorem. Applications of these spaces to the Cauchy problem in Volume 3 will illustrate the well-known statement of Hadamard on the relation between uniqueness theorems in the Cauchy problem on the one hand, and the theory of quasi-analytic functions and the general theory of functions of a complex variable, on the other. Spaces of type $S$ yield natural limits for a sufficiently flexible Fourier transform theory because these spaces go over into each other under Fourier transformation; hence, Chapter IV is a natural continuation of Chapter III, devoted to Fourier transforms. Moreover, various operators of the form $f(d/dx)$, where $f(t)$ is an entire function, can be constructed in spaces of type $S$, and are also applicable to generalized functions. The Fourier transforms of generalized functions, considered as continuous linear functionals on spaces of type $S$ and $W$, as well as the construction of operators of the form $f(d/dx)$, applicable to the generalized functions, are indeed the fundamental tools which we shall use in Volume 3 for studying the Cauchy problem.

In order not to overburden the exposition here, we have referred a summary of the results referring to spaces of type $W$ to an appendix; proofs of these results are collected in Chapter I of Volume 3. The reader interested only in problems of eigenfunction expansions may turn to Chapter IV of Volume 3 directly after having completed Chapters I and II of the present volume.

The authors take this opportunity to express their heartfelt gratitude to all their colleagues who assisted in writing this volume. To D. A. Raikov we owe a number of essential improvements in the first chapter. B. Ya. Levin constructed the proof of some necessary theorems from the theory of entire functions (Chapter IV) at our request. G. N. Zolotarev indicated some simplifications in the exposition of Chapters II and III. The section on the Hilbert transform (Chapter III) was written according to an idea of N. Ya. Vilenkin. Finally, a multitude of improvements has been inserted in accordance with suggestions of M. S. Agranovich, who edited the entire text of this volume.

*Moscow, 1958*

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G. E. SHILOV

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CHAPTER I

# LINEAR TOPOLOGICAL SPACES

## 1. Definition of a Linear Topological Space[1]

### 1.1. System of Axioms for a Linear Topological Space

A collection $\Phi$ of elements $\varphi, \psi, \ldots$ is called a *linear topological space* if the following conditions are fulfilled:

**I.** $\Phi$ is a linear space with multiplication by (real or) complex numbers.

**II.** $\Phi$ is a topological space.

**III.** The operations of addition and multiplication by numbers are continuous relative to the topology of $\Phi$.

Let us consider each of these conditions in detail.

**I.** *The collection $\Phi$ is a linear space with multiplication by complex numbers.*

This means that an operation of addition of elements in $\Phi$, and an operation of multiplying elements by (complex) numbers $\lambda, \mu, \ldots$ is defined, and the following axioms are fulfilled:

**I.1.** $\varphi + \psi = \psi + \varphi$ (*addition is commutative*);

**I.2.** $\varphi + (\psi + \chi) = (\varphi + \psi) + \chi$ (*addition is associative*);

**I.3.** *There is an element* $0$ *such that* $\varphi + 0 = \varphi$ *for any* $\varphi$;

**I.4.** *For every element* $\varphi$, *there is an element* $\psi$ *such that* $\varphi + \psi = 0$ (*the negative element*);

**I.5.** $1 \cdot \varphi = \varphi$ for any $\varphi \in \Phi$;

**I.6.** $\lambda(\mu\varphi) = (\lambda\mu)\varphi$;

[1] Since Section 1 is of a preparatory nature, the reader who is familiar with the definition of a linear topological space can proceed directly to Section 2, and return to Section 1 when necessary.

**I.7.** $(\lambda + \mu)\varphi = \lambda\varphi + \mu\varphi$;

**I.8.** $\lambda(\varphi + \psi) = \lambda\varphi + \lambda\psi$.

Axiom I.6 denotes the associativity of multiplication by numbers; Axioms I.7 and I.8 express two laws of distributivity related to addition and multiplication by numbers.

It can be deduced, in turn, from Axioms I.1–I.8 that the product of 0 with any element $\varphi$ is the element 0, and that the product of the number $-1$ with any element $\varphi$ is the negative of $\varphi$, which is therefore naturally denoted by $-\varphi$.

We present some simple definitions pertaining to linear spaces.

The collection of all sums $\varphi + \psi$, where $\varphi$ ranges over a set $A$ in the linear space $\Phi$, is called the *translate of the set $A$ by the vector $\psi$.*

The collection of all sums $\varphi + \psi$, where $\varphi$ ranges over a set $A$, and $\psi$ ranges over a set $B$, is called the *sum* (more precisely, the *arithmetic sum*) of the sets $A$ and $B$, and is denoted by $A + B$. The *arithmetic difference $A - B$* is defined analogously.

The collection of all products of the elements $\varphi$ of a set $A$ by a number $\lambda$ is called the *$\lambda$-tuple* (or *$\lambda$-dilation*) *of the set $A$* and is denoted by $\lambda A$. (We remark that in general $2A \neq A + A$.) In particular, $-A$ is the collection of all the negatives of elements in $A$.

**II.** *The collection $\Phi$ is a topological space.*

This means that a system $\{U\}$ of subsets of $\Phi$, called (open) *neighborhoods*, is specified, and the following axioms are satisfied:

**II.1.** *Every point $\varphi \in \Phi$ belongs to some neighborhood $U = U(\varphi)$*;

**II.2.** *If a point $\varphi$ belongs to neighborhoods $U$ and $V$, then it belongs to a neighborhood $W$ which lies entirely in the intersection of $U$ with $V$*;

**II.3.** *For any pair of points $\varphi \neq \psi$, there is some neighborhood $U$ which contains $\varphi$ but does not contain $\psi$.*

The neighborhoods and all of their unions (finite and infinite) form the system of *open sets*. An open set is characterized by the fact that *every one of its points is an interior point*, i.e., it belongs to a neighborhood which lies in the given set. It is easy to obtain from this that *the union of any number of open sets and the intersection of any finite number of open sets are open sets.*

Henceforth, by a *neighborhood of a given point* we will understand any neighborhood containing the point.

A point $\varphi_0$ is called an *adherence point of a set $A$*, if every neighborhood of $\varphi_0$ contains a point of $A$. In particular, every point of a set $A$ is an adherence point of $A$. There are two possibilities for adherence points of a set $A$:

1. There exists a neighborhood of $\varphi_0$ which contains only a finite number of distinct points of $A$;
2. Every neighborhood of $\varphi_0$ contains an infinite number of distinct points of $A$.

In the first case, using Axioms II.2 and II.3, one can construct a neighborhood of $\varphi_0$ which contains no point of $A$ other than $\varphi_0$ itself. In this case $\varphi_0$ belongs to $A$, and is called an *isolated point* of $A$.

In the second case $\varphi_0$ is called a *limit point* of $A$. An isolated adherence point of $A$ always belongs to $A$; a limit point may or may not belong to $A$.

The collection of all adherence points of $A$ forms the *closure* $\bar{A}$ of $A$; thus, the closure of a set $A$ is obtained by adjoining to $A$ those of its limit points which do not belong to it.

A set $A$ is said to be *closed* if it contains all of its adherence points. One can verify that the closure of any set is closed. The closed sets can be characterized as the complements (with respect to all of $\Phi$) of the open sets. It follows that the union of a finite number of closed sets and the intersection of any number of closed sets are closed sets.

A set $A$ is said to be *dense* in the space $\Phi$ (more precisely, *everywhere dense*) if its closure coincides with $\Phi$. Example: the set of rational points on the line.

A set is said to be *nowhere dense*, if its closure has no interior point. Example: the Cantor set on the line.

The collection of all open and all closed sets of a space $\Phi$ forms its *topology*.

One can arrive at the same topology in a space (i.e., the same system of open and closed sets), starting from two different systems of neighborhoods. For example, in defining the natural topology on the real line we can, on the one hand, take as neighborhoods all intervals with rational endpoints and, on the other hand, all intervals with irrational endpoints. We will call different systems of neighborhoods *equivalent*, if they lead to the same topology. The following simple condition is both necessary and sufficient for the equivalence of two given neighborhood systems $\{U\}$ and $\{V\}$: Every neighborhood $U$ contains a neighborhood $V$, and every neighborhood $V$ contains a neighborhood $U$.

**Convergent Sequences.** A sequence $\varphi_1, \varphi_2, \ldots, \varphi_\nu, \ldots$ of elements of a topological space $\Phi$ is said to *converge to an element* $\varphi$, if each neighborhood of $\varphi$ contains all the points of the sequence, starting with some given one whose index in general depends upon the neighborhood.

In this case, one writes $\varphi = \lim_{\nu\to\infty} \varphi_\nu$. On the line (or in $m$-dimensional space), every limit point of a given set $A$ is the limit of some convergent *sequence* of points belonging to $A$. The assumption, natural at first glance, that in the general case also, every limit point of a set $A$ must be the limit of some (countable) sequence $\varphi_\nu \in A$ ($\nu = 1, 2,\ldots$) turns out, under closer examination, to be false.

Example. Let us consider the collection $\Phi$ of (all) bounded real functions $\varphi(x)$ on the interval $0 \leqslant x \leqslant 1$, with the ordinary linear operations; we define a neighborhood $U = U(\varphi_0 ; x_1 ,\ldots, x_n ; \epsilon)$ of a given element $\varphi = \varphi_0(x)$ by specifying a finite number of points $x_1 ,\ldots, x_n$ and a number $\epsilon > 0$; this neighborhood consists of all $\varphi \in \Phi$ for which $|\varphi(x_j) - \varphi_0(x_j)| < \epsilon$, $j = 1,\ldots, n$. We form the set $A$ of functions $\varphi(x)$, each of which equals 1 everywhere, with the exception of a finite number of points at which it equals 0. Obviously, $\varphi_0(x) \equiv 0$ is an adherence point of $A$. At the same time, no (countable) sequence of elements $\{\Psi_\nu(x)\}$ of $A$ can converge to zero, since, in view of the uncountability of the continuum $0 \leqslant x \leqslant 1$, one can always find a point $x_0$ at which all of the $\varphi_\nu(x)$ equal 1, and consequently no one of them lies within any neighborhood of the form $U(\varphi_0 ; x_0 ; \frac{1}{2})$.

Of course, it would be very helpful in analysis if any limit point $\varphi_0$ of every set $A$ were always the limit of some sequence of points of $A$. This property holds in topological spaces in which an additional condition is satisfied:

**The first axiom of countability at a point $\varphi_0$.** *The point $\varphi_0$ has a countable neighborhood basis.*

A system $\{U\}$ of neighborhoods $U_1$, $U_2$,... of $\varphi_0$ is said to be a *basis of the neighborhoods of* $\varphi_0$, if every neighborhood $V$ of $\varphi_0$ contains at least one of the $U_\nu$.

Let us show that *if the first axiom of countability is satisfied at a point $\varphi_0$, then from any set $A$ which has $\varphi_0$ as a limit point, it is possible to select a sequence $\varphi_1$, $\varphi_2$,... which converges to $\varphi_0$.*

First we note that we can always consider a countable neighborhood basis to be decreasing, so that $U_1 \supset U_2 \supset \cdots$; indeed, if this condition is not fulfilled, then in place of $U_2$, we take a neighborhood $U_2'$ lying in the intersection of $U_1$ and $U_2$; in place of $U_3$, we take a neighborhood $U_3'$ lying in the intersection of $U_2'$ and $U_3$, and so on. Let us now assume that $\varphi_0$ is an adherence point of some set $A$. We choose a point $\varphi_\nu \in A$ in each neighborhood $U_\nu$ (assuming that $U_1 \supset U_2 \supset \cdots$); then

$$\varphi_0 = \lim_{\nu\to\infty} \varphi_\nu .$$

Indeed, for any neighborhood $V$ of $\varphi_0$, there is some neighborhood $U_\nu \subset V$, and since $U_{\nu+p} \subset U_\nu$, it follows that $\varphi_{\nu+p} \in U_{\nu+p} \subset V$ for any $p$; thus, any neighborhood of $\varphi_0$ contains all the points $\varphi_1, \varphi_2, \ldots$ starting with some one, as was required.

If the first axiom of countability is fulfilled at every point of the space $\Phi$, then one says that *it is fulfilled everywhere in* $\Phi$.

We now proceed to condition III.

**III.** *The operations of addition and multiplication by a number are continuous relative to the topology of* $\Phi$.

Conditions I and II, which we have considered in detail just above, described separate properties of the linear operations and the operation of passing to a limit; condition III establishes the connection between these. Condition III may be divided into the following two axioms.

**III.1.** *The continuity of addition and subtraction. If one of the relations*

$$\varphi_0 \pm \psi_0 = \chi_0$$

*holds, then for any neighborhood* $U$ *of* $\chi_0$ *there is a neighborhood* $V$ *of* $\varphi_0$ *and a neighborhood* $W$ *of* $\psi_0$ *such that* $\varphi \in V$ *and* $\psi \in W$ *imply* $\varphi \pm \psi \in U$ *(or, briefly,* $V \pm W \subset U$*).*

**III.2.** *The continuity of multiplication by a number. If* $\lambda_0\varphi_0 = \psi_0$, *then for any neighborhood* $U$ *of* $\psi_0$ *there is a neighborhood* $V$ *of* $\varphi_0$ *and a number* $\epsilon > 0$ *such that* $\varphi \in V$ *and* $|\lambda - \lambda_0| < \epsilon$ *imply* $\lambda\varphi \in U$.

Let us first consider some consequences of Axiom III.1.

First of all, we note that *the collection of all translates of all the neighborhoods of* 0 *defines a system of neighborhoods in* $\Phi$ *which is equivalent to the original system*. Indeed, that this collection is actually a collection of neighborhoods (i.e., satisfies II.1–II.3) is easy to show. To see that this system is equivalent to the original system of neighborhoods, let $V = \varphi_0 + U$, where $U$ is a neighborhood of 0. If $\varphi \in V$, then $\varphi - \varphi_0 \in U$. Thus we can find neighborhoods $W_1$ and $W_2$ of $\varphi$ and $\varphi_0$, respectively, such that $W_1 - W_2 \subset U$. In particular, since $\varphi_0 \in W_2$, we have $W_1 - \varphi_0 \subset U$, or $W_1 \subset \varphi_0 + U = V$. Conversely, given a neighborhood $U$ of a point $\varphi_0$, since $\varphi_0 + 0 = \varphi_0$, we can find a neighborhood $W_1$ (of $\varphi_0$) and a neighborhood $W_2$ (of 0) such that $W_1 + W_2 \subset U$. Since $\varphi_0 \in W_1$, we have $V = \varphi_0 + W_2 \subset U$, as was required.

Thus, *the topology in* $\Phi$ *can be reconstructed from the system of neighborhoods of zero*; subjecting them to all possible translations, we obtain a complete system of neighborhoods for the entire space. This means that from the topological point of view, the structure of the space is the same at all points; a simple translation (taking the entire space onto itself) carries any point into any other, and every neighborhood of the first point is carried into a neighborhood of the second. The local properties of the topology of the space at one point are the same as at every other point. In particular, if the first axiom of countability is fulfilled at one point, for example at 0, then it is fulfilled at every other point, i.e., it is fulfilled over the entire space.

We remark, further, that *a linear topological space is always regular*, i.e., for any point $\varphi$ and neighborhood $U$ of $\varphi$ there is a smaller neighborhood $V$ of $\varphi$ which lies in $U$, together with its closure.

For the proof it is sufficient, in view of the homogeneity of the topology in $\Phi$, to consider the case $\varphi = 0$. In view of the continuity of subtraction, we can find two neighborhoods $W_1$ and $W_2$ of 0 such that $W_1 - W_2 \subset U$, and if we further take a neighborhood $W$ of 0 lying in the intersection of $W_1$ and $W_2$, then we will have $W - W \subset U$. We assert that the closure $\overline{W}$ of $W$ lies in $U$.

In fact, let $\psi$ be an adherence point of $W$; then the neighborhood $V = \psi + W$ of the point $\psi$ contains some point of $W$. Suppose that $\chi \in V \cap W$, so that $\chi = \psi + \varphi$, where $\varphi \in W$. Then

$$\psi = \chi - \varphi \in W - W \subset U,$$

as was asserted.

Let us now turn to those properties of linear topological spaces which are related to the continuity of multiplication by a number. (Axiom III.2.)

First of all, we shall show that *any dilation $\lambda U$, $\lambda \neq 0$ of an open set $U$ is an open set*. Indeed, let $\psi = \lambda\varphi$, where $\varphi \in U$; then $\varphi = (1/\lambda)\psi$ and, given a neighborhood $V$ of $\varphi$, we can find a neighborhood $W$ of $\psi$ such that $(1/\lambda)\, W \subset V$, or $W \subset \lambda V$. Taking $V \subset U$, we see that $W \subset \lambda V \subset \lambda U$, i.e., the point $\psi$ lies in $\lambda U$ together with its neighborhood $W$, as was required.

In particular, every dilation $\lambda U$, $\lambda \neq 0$ of a neighborhood $U$ of zero is a neighborhood of zero, and if $\lambda \neq 0$ is fixed, then the collection of sets of the form $\lambda U$, where $U$ ranges over a basis of the neighborhoods of zero, is itself a basis of the neighborhoods of zero. It is sufficient to show that for any neighborhood $V$ of zero, one can find a set $U$ in the basis of the neighborhoods of zero for which $\lambda U \subset V$. But the existence of such a $U$ follows at once from the continuity of multiplication by $\lambda$ and the definition of a neighborhood basis.

We can use the neighborhoods of the form $\lambda U$ to construct certain special systems of neighborhoods of zero, called *normal* neighborhoods.