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602

Recent Developments in Algebraic and Combinatorial Aspects of Representation Theory

International Congress of Mathematicians Satellite Conference
Algebraic and Combinatorial Approaches to Representation Theory
August 12–16, 2010
National Institute of Advanced Studies, Bangalore, India
Follow-up Conference
May 18–20, 2012
University of California, Riverside, CA

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Recent Developments in Algebraic and Combinatorial Aspects of Representation Theory

Preface

Representation theory of Lie algebras, quantum groups and algebraic groups represent a major area of mathematical research in the twenty-first century with numerous applications in other areas of mathematics (geometry, number theory, combinatorics, finite and infinite groups, etc.) and mathematical physics (such as conformal field theory, statistical mechanics, integrable systems). Current research topics in representation theory include quantized enveloping and function algebras, Kac-Moody Lie algebras, Hecke algebras, canonical bases and crystal bases, vertex operator algebras, Hall algebras, A_∞-algebras, quivers, cluster algebras, Hopf algebras, and Artin-Schelter regular algebras. In particular, representation theory of quantized Kac-Moody Lie algebras and cohomological theories in noncommutative algebraic geometry have taken the lead not only within research areas in algebra but also in other areas of mathematics and physics.

There are various approaches to study representation theory. This proceedings is based on two conferences which focused on the algebraic and combinatorial approaches to the representation theory.

The first conference was held at the National Institute of Advanced Studies, Bangalore, India during August 12-16, 2010. This was a satellite conference preceding the International Congress of Mathematicians held at Hyderabad, India. The second follow-up conference was held at the University of California, Riverside during May 18-20, 2012. The speakers at each of these conferences were invited to contribute to this proceedings.

The articles in this proceedings touch upon a broad spectrum of topics including quantum groups, crystal bases, categorification, toroidal algebras, vertex algebras, Hecke algebras, Kazhdan-Lusztig bases, etc.

We thank all participants and speakers at both of these conferences for their participation and valuable contributions. In particular we thank the speakers who contributed to this proceedings volume. We are grateful to the National Science Foundation, USA and the International Center for Theoretical Sciences, India for providing the financial support for these conferences. Finally, we appreciate the help from the National Institute of Advanced Studies, Bangalore and the University of California, Riverside during the respective conferences.

The Editors

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Kostka systems and exotic t-structures for reflection groups

Pramod N. Achar

Abstract. Let W be a complex reflection group, acting on a complex vector space $\mathfrak{h}$. Kato has recently introduced the notion of a "Kostka system," which is a certain collection of finite-dimensional W-equivariant modules for the symmetric algebra on $\mathfrak{h}$. In this paper, we show that Kostka systems can be used to construct "exotic" t-structures on the derived category of finite-dimensional modules, and we prove a derived-equivalence result for these t-structures.

1. Introduction

1.1. Overview. In the early 1980's, Shoji [**S1**, **S2**] and Lusztig [**L3**] showed that Green functions—certain polynomials arising in the representation theory of finite groups of Lie type—can be computed by a rather elementary procedure, now often known as the *Lusztig–Shoji algorithm.* This algorithm can be interpreted as a computation in the Grothendieck group of the derived category of mixed ℓ-adic complexes on the nilpotent cone of a reductive algebraic group, with the simple perverse sheaves playing a key role; see [**A3**].

In recent work [**K1**], Kato has proposed an alternative interpretation of Green functions in terms of the Grothendieck group of the (derived) category of graded modules over the ring $\mathbf{A}_W = \mathbb{C}[W] \# \mathbb{C}[\mathfrak{h}^*]$, where W is the Weyl group, and $\mathfrak{h}$ is the Cartan subalgebra. In place of simple perverse sheaves, the key objects are now projective $\mathbf{A}_W$-modules. Thus, Kato's viewpoint is "Koszul dual" to the geometric one. A prominent place is given to certain collections of finite-dimensional $\mathbf{A}_W$-modules (denoted by K_χ in [**K1**] and by $\bar{\nabla}_\chi$ here), called *Kostka systems.*

In this paper, we study Kostka systems as generators of the derived category $D^{\mathrm{b}}_{\mathrm{fd}}(\mathbf{A}_W)$ of finite-dimensional $\mathbf{A}_W$-modules. We prove that they form a dualizable quasi-exceptional sequence, which implies that they determine a new t-structure on $D^{\mathrm{b}}_{\mathrm{fd}}(\mathbf{A}_W)$, called the *exotic* t-structure. The heart of this t-structure, denoted by $\mathcal{E}x_W$, is a finite-length weakly quasi-hereditary category. The main result (see Theorem 6.9) states that there is an equivalence of triangulated categories

$$D^{\mathrm{b}}\mathcal{E}x_W \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{fd}}(\mathbf{A}_W). \tag{1.1}$$

Of course, projective $\mathbf{A}_W$-modules cannot belong to $\mathcal{E}x_W$, since they are not finite-dimensional. Nevertheless, in some ways, they behave as though they were tilting objects of $\mathcal{E}x_W$. Thus, in a loose sense, which we do not attempt to make precise

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geometric Langlands duality	*Springer theory*
perverse sheaves on the affine Grassmannian of G; geometric Satake	perverse sheaves on the nilpotent cone of G; Springer correspondence
$G\times\mathbb{G}_{\mathrm{m}}$-equivariant coherent sheaves on the dual Lie algebra $\check{\mathfrak{g}}$	graded $\mathbf{A}_W$-modules, or $W\times\mathbb{G}_{\mathrm{m}}$-equivariant coherent sheaves on $\check{\mathfrak{h}}$
coherent sheaves supported on the dual nilpotent cone $\check{\mathcal{N}}\subset\check{\mathfrak{g}}$	finite-dimensional $\mathbf{A}_W$-modules, or coherent sheaves supported on $\{0\}\subset\check{\mathfrak{h}}$
Andersen–Jantzen sheaves on $\check{\mathcal{N}}$	Kostka systems $\{\bar{\nabla}_\chi\}$
exotic (or perverse-coherent) t-structure on $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\check{\mathcal{N}})$	exotic t-structure on $D^{\mathrm{b}}_{\mathrm{fd}}(\mathbf{A}_W)$

TABLE 1. Geometric Langlands duality and Springer theory

in this paper, the category $\mathcal{E}x_W$ can be thought of as "Ringel dual" to the category of $\mathbf{A}_W$-modules. (See Section 6.3.)

1.2. Analogy with geometric Langlands duality. A theme arising in geometric Langlands duality is that perverse or constructible sheaves on a (partial) affine flag variety for a reductive group G should be described in terms of coherent sheaves on varieties related to the dual group $\check{G}$. For instance, the spherical equivariant derived category of the affine Grassmannian Gr is closely related to coherent sheaves on the dual Lie algebra $\check{\mathfrak{g}}$; see [**BF**].

Springer theory is a rich source of phenomena that seem to be "shadows at the level of the Weyl group" of geometric Langlands duality. Indeed, the Springer correspondence itself is in part a Weyl-group shadow of the geometric Satake equivalence [**AH**, **AHR**]. Another example is Rider's equivalence [**Rid**] relating the equivariant derived category of the nilpotent cone to $\mathbf{A}_W$-modules, or, equivalently, to W-equivariant coherent sheaves on the dual Cartan subalgebra $\check{\mathfrak{h}}$: this resembles the aforementioned result of [**BF**]. Further parallels are summarized in Table 1.

Kato's results and those of the present paper are contributions to the study of the "Galois side" (or "coherent side") of this picture. Among (complexes of) coherent sheaves on $\check{\mathfrak{g}}$, those supported on the dual nilpotent cone $\check{\mathcal{N}}$ are of particular importance, especially those in the heart of an exotic t-structure determined by the so-called Andersen–Jantzen sheaves [**B1**, **B2**]. The Weyl-group analogue should involve sheaves supported on $\{0\}\subset\check{\mathfrak{h}}$—in other words, finite-dimensional $\mathbf{A}_W$-modules. Specifically, Kostka systems should be thought of as Weyl-group analogues of Andersen–Jantzen sheaves, and the equivalence (1.1) as a Weyl-group shadow of the derived equivalences from [**B2**] or [**A4**, Theorem 1.2].

1.3. Green functions for complex reflection groups. The Lusztig–Shoji algorithm itself only requires knowing the reflection group W and the preorder $\precsim$ on $\mathrm{Irr}(W)$ induced by the Springer correspondence. (See [**A1**].) In particular, it makes sense to carry out the algorithm with a different, "artificial" preorder, or even with W replaced by a complex reflection group that is not the Weyl group of any algebraic group. See [**S3, S4, GM**] for variations and conjectures on the Lusztig–Shoji algorithm.

One of Kato's aims in [**K1**] was to provide a categorical framework for interpreting the output of the algorithm in this more general setting, where geometric tools like perverse sheaves are not available. In the present paper, we try to preserve this goal. Most definitions and constructions in this paper make sense for arbitrary complex reflection groups and arbitrary preorders on $\mathrm{Irr}(W)$. We do invoke some results of Kato whose proofs involve the geometry of the nilpotent cone, and are thus valid only for Weyl groups. However, outside of Section 4, we treat these results as axioms: if, in the future, non-geometric proofs of these results become available for other complex reflection groups, then the main results of this paper will extend to those complex reflection groups as well.

1.4. Acknowledgments. The author is grateful to Syu Kato for a number of helpful comments. This paper has, of course, been deeply influenced by his ideas. I would also like to thank the organizers of the ICM 2010 satellite conference on Algebraic and Combinatorial Approaches to Representation Theory for having given me the opportunity to participate.

2. Notation and preliminaries

2.1. Graded rings and vector spaces. If R is a noetherian graded $\mathbb{C}$-algebra, we write R-gmod (resp. R-$\mathrm{gmod}_{\mathrm{fd}}$) for the category of finitely-generated (resp. finite-dimensional) graded left R-modules. For any $M \in R$-gmod, we write $\mathrm{gr}_k V$ for its k-th graded component. We define $M\langle 1\rangle$ to be the new graded module with

$$\mathrm{gr}_k(M\langle 1\rangle) = \mathrm{gr}_{k-1} M.$$

The operation $M \mapsto M\langle 1\rangle$ also makes sense for chain complexes of modules over R. If M and N are (complexes of) graded R-modules, we define $\underline{\mathrm{Hom}}_R(M,N)$ (or simply $\underline{\mathrm{Hom}}(M,N)$) to be the graded vector space given by

$$\mathrm{gr}_k \underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}(M, N\langle -k\rangle).$$

We use the term *grade* to refer to the integers k such that $\mathrm{gr}_k M \neq 0$, reserving the term *degree* for homological uses, such as indexing the terms in a chain complex. Thus, a module M is said to *have grades* $\geq n$ if $\mathrm{gr}_k M = 0$ for all $k < n$. If M is a chain complex of modules, we say that M *has grades* $\geq n$ if all its cohomology modules $H^i(M)$ have grades $\geq n$.

If M and N are objects in a derived category of R-modules, we employ the usual notation $\mathrm{Hom}^i(M,N) = \mathrm{Hom}(M, N[i])$, as well as $\underline{\mathrm{Hom}}^i(M,N) = \underline{\mathrm{Hom}}(M,N[i])$.

2.2. Reflection groups and phyla. Throughout the paper, W will be a fixed complex reflection group, acting on a finite-dimensional complex vector space $\mathfrak{h}$. Let $S\mathfrak{h}$ be the symmetric algebra on $\mathfrak{h}$, regarded as a graded ring by declaring elements of $\mathfrak{h} \subset S\mathfrak{h}$ to have degree 1. Our main object of study is the ring

$$\mathbf{A}_W = \mathbb{C}[W] \# S\mathfrak{h}.$$

Let $\mathbf{A}_W$-gmod be the category of finitely-generated graded $\mathbf{A}_W$-modules. Henceforth, all $\mathbf{A}_W$-modules are assumed to be objects of $\mathbf{A}_W$-gmod.

Let $\mathrm{Irr}(W)$ denote the set of irreducible complex characters of W. For $\chi \in \mathrm{Irr}(W)$, let $\bar{\chi}$ denote the complex-conjugate character. If W is a Coxeter group, then all characters are real-valued, and $\bar{\chi} = \chi$, but general complex reflection groups may have characters that are not real-valued.

We also assume throughout that $\mathrm{Irr}(W)$ is equipped with a fixed total preorder $\precsim$, and that the equivalence relation $\sim$ induced by this preorder satisfies

$$\chi \sim \bar{\chi}$$

for all $\chi \in \mathrm{Irr}(W)$. (In [**K1**], a preorder satisfying this condition is said to be *of Malle type.* Many arguments in this paper can likely be adapted to the case where this condition is dropped, but these generalizations will not be pursued here.) Following [**A1**], the equivalence classes for $\sim$ are called *phyla.* For $\chi \in \mathrm{Irr}(W)$, we write $[\chi]$ for the phylum to which it belongs.

2.3. $\mathbf{A}_W$-modules. For each $\chi \in \mathrm{Irr}(W)$, choose a representation L_χ giving rise to that character. Consider the vector space

$$P_\chi = L_\chi \otimes \mathrm{S}\mathfrak{h}.$$

We regard this as a graded $\mathbf{A}_W$-module by having $\mathrm{S}\mathfrak{h}$ act on the second factor, and having W act on both factors. This is a projective $\mathbf{A}_W$-module, and every indecomposable projective in $\mathbf{A}_W$-gmod is of the form $P_\chi\langle n\rangle$ for some χ and some n. See [**K1**, Lemma 2.2].

For brevity, we write $D^{\mathrm{b}}(\mathbf{A}_W)$ rather than $D^{\mathrm{b}}(\mathbf{A}_W\text{-gmod})$ for the bounded derived category of $\mathbf{A}_W$-gmod, and likewise for $D^-(\mathbf{A}_W)$ and $D^+(\mathbf{A}_W)$.

We will occasionally need to consider groups

$$\mathrm{Hom}(M, N) \qquad \text{with } M \in D^+(\mathbf{A}_W) \text{ and } N \in D^-(\mathbf{A}_W). \tag{2.1}$$

This is to be understood by identifying $D^+(\mathbf{A}_W)$ and $D^-(\mathbf{A}_W)$ with full subcategories of the unbounded derived category $D(\mathbf{A}_W)$. Because $\mathbf{A}_W$ has finite global dimension, we can ignore some of the technical difficulties that usually arise with unbounded derived categories. In particular, according to [**AF**, Proposition 3.4], complexes of projective modules in $D(\mathbf{A}_W)$ are homotopy-projective. Moreover, every object in $D^+(\mathbf{A}_W)$ is isomorphic to a bounded-below complex of projectives; see [**AF**, §1.6]. Thus, if M and N are both given by explicit complexes of projectives, then (2.1) is simply the set of homotopy classes of chain maps between those complexes.

2.4. Duality. For $M \in \mathrm{S}\mathfrak{h}$-gmod, the graded vector space $\underline{\mathrm{Hom}}_{\mathrm{S}\mathfrak{h}}(M, \mathrm{S}\mathfrak{h})$ can naturally be regarded as an object of $\mathrm{S}\mathfrak{h}$-gmod itself. It is well known that the derived functor $\mathbb{D} = R\underline{\mathrm{Hom}}_{\mathrm{S}\mathfrak{h}}(-, \mathrm{S}\mathfrak{h})$ gives an equivalence of categories $D^-(\mathrm{S}\mathfrak{h})^{\mathrm{op}} \xrightarrow{\sim} D^+(\mathrm{S}\mathfrak{h})$; see [**H**, Example V.2.2]. Moreover, $\mathbb{D}$ takes bounded complexes to bounded complexes, and so gives an antiautoequivalence of $D^{\mathrm{b}}(\mathrm{S}\mathfrak{h})$.

Now, suppose that $M \in \mathbf{A}_W$-gmod. Then the $\mathrm{S}\mathfrak{h}$-module $\underline{\mathrm{Hom}}_{\mathrm{S}\mathfrak{h}}(M, \mathrm{S}\mathfrak{h})$ carries an obvious W-action, and so can be regarded as an object of $\mathbf{A}_W$-gmod. From the facts above about $\mathbb{D}$, one can deduce the W-equivariant analogues: there is an equivalence of categories

$$\mathbb{D} = R\underline{\mathrm{Hom}}_{\mathrm{S}\mathfrak{h}}(-, \mathrm{S}\mathfrak{h}) : D^-(\mathbf{A}_W)^{\mathrm{op}} \xrightarrow{\sim} D^+(\mathbf{A}_W)$$

that restricts to an equivalence $D^{\mathrm{b}}(\mathbf{A}_W)^{\mathrm{op}} \xrightarrow{\sim} D^{\mathrm{b}}(\mathbf{A}_W)$. In particular, we have

$$\mathbb{D}(P_\chi) \cong P_{\bar{\chi}}. \tag{2.2}$$

2.5. Finite-dimensional modules. As noted in the introduction, the main results of this paper involve the category

$$D^{\mathrm{b}}_{\mathrm{fd}}(\mathbf{A}_W) = \left\{ X \in D^{\mathrm{b}}(\mathbf{A}_W) \;\middle|\; \begin{array}{c}\text{for all } i,\ H^i(X) \text{ is a} \\ \text{finite-dimensional } \mathbf{A}_W\text{-module}\end{array} \right\}.$$

We will occasionally make use of the fact that this is equivalent to the derived category $D^{\mathrm{b}}(\mathbf{A}_W\text{-gmod}_{\mathrm{fd}})$. That fact is an instance of the following lemma.

LEMMA 2.1. *Let $R = \bigoplus_{n\geq 0} R_n$ be a nonnegatively graded noetherian $\mathbb{C}$-algebra, and assume that R_0 is finite-dimensional. Then the natural functor*

$$D^{\mathrm{b}}(R\text{-gmod}_{\mathrm{fd}}) \to D^{\mathrm{b}}(R\text{-gmod})$$

is fully faithful.

PROOF. We begin with a digression. Since R is noetherian and R_0 is finite-dimensional, each R_n must be finite-dimensional. It follows that for any $M \in R\text{-gmod}$, each $\mathrm{gr}_n M$ is finite-dimensional. Now, given $k \in \mathbb{Z}$, let $M_{\geq k} \subset M$ be the submodule generated by all homogeneous elements of grade $\geq k$, and let $M_{\leq k} = M/M_{\geq k+1}$. It is easy to see that the functors $M \mapsto M_{\geq k}$ and $M \mapsto M_{\leq k}$ are exact. Moreover, $M_{\leq k}$ is always finite-dimensional.

Returning to the statement of the lemma, recall that by a standard argument (see [**BBD**, Proposition 3.1.16]), the question can be reduced to showing that the following natural morphism of δ-functors (for $A, B \in R\text{-gmod}_{\mathrm{fd}}$) is an isomorphism:

$$\mathrm{Ext}^i_{R\text{-gmod}_{\mathrm{fd}}}(A,B) \to \mathrm{Ext}^i_{R\text{-gmod}}(A,B). \tag{2.3}$$

When $i = 0$, this is obvious, and for $i = 1$, this follows from the fact that $R\text{-gmod}_{\mathrm{fd}}$ is a Serre subcategory of $R\text{-gmod}$.

For general $i > 0$, each element of $\mathrm{Ext}^i_{R\text{-gmod}}(A,B)$ is represented by some exact sequence

$$0 \to B \to M^i \to M^{i-1} \to \cdots \to M^1 \to A \to 0. \tag{2.4}$$

Since A and B are finite-dimensional, there is a k such that $A_{\geq k+1} = B_{\geq k+1} = 0$. Applying the exact functor $M \mapsto M_{\leq k}$ to (2.4) gives an exact sequence

$$0 \to B \to M^i_{\leq k} \to M^{i-1}_{\leq k} \to \cdots \to M^1_{\leq k} \to A \to 0. \tag{2.5}$$

This represents the same element of $\mathrm{Ext}^i_{R\text{-gmod}}(A,B)$ as (2.4), but since every term is finite-dimensional, it also represents an element of $\mathrm{Ext}^i_{R\text{-gmod}_{\mathrm{fd}}}(A,B)$. We have just shown that (2.3) is surjective for all i.

According to [**BBD**, Remarque 3.1.17(1)], if (2.3) failed to be an isomorphism for some i, then for a minimal such i, it would be injective but not surjective. So (2.3) is indeed an isomorphism for all i. □

2.6. Admissible subcategories of triangulated categories. We conclude this section with a review of a result from homological algebra that we will use a number of times in the sequel.

DEFINITION 2.2. Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{A}$ and $\mathcal{B}$ be two full triangulated subcategories. We say that $(\mathcal{A}, \mathcal{B})$ is an *admissible pair* if the following two conditions hold:

(1) We have $\mathrm{Hom}(A,B) = 0$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
(2) Together, the objects in $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{D}$ as a triangulated category.

This is slightly nonstandard terminology: usually, $\mathcal{A}$ is said to be *right-admissible* if there exists a $\mathcal{B}$ such that the conditions above hold; dually, $\mathcal{B}$ is said to be *left-admissible*. The following lemma collects some consequences and equivalent characterizations.

LEMMA 2.3 ([**BK**, Propositions 1.5 and 1.6]). *Let $(\mathcal{A}, \mathcal{B})$ be an admissible pair in a triangulated category $\mathcal{D}$. Then:*

(1) *The inclusion $\mathcal{A} \to \mathcal{D}$ admits a right adjoint $\imath : \mathcal{D} \to \mathcal{A}$.*
(2) *The inclusion $\mathcal{B} \to \mathcal{D}$ admits a left adjoint $\jmath : \mathcal{D} \to \mathcal{B}$.*
(3) *For every $X \in \mathcal{D}$, there is a functorial distinguished triangle*
$$\imath(X) \to X \to \jmath(X) \to .$$
(4) *We have $\mathcal{A} = \{X \in \mathcal{D} \mid \mathrm{Hom}(X, B) = 0$ for all $B \in \mathcal{B}\}$.*
(5) *We have $\mathcal{B} = \{X \in \mathcal{D} \mid \mathrm{Hom}(A, X) = 0$ for all $A \in \mathcal{A}\}$.*
(6) *The inclusions $\mathcal{A} \to \mathcal{D}$ and $\mathcal{B} \to \mathcal{D}$ induce equivalences of triangulated categories*
$$\mathcal{A} \xrightarrow{\sim} \mathcal{D}/\mathcal{B} \qquad \text{and} \qquad \mathcal{B} \xrightarrow{\sim} \mathcal{D}/\mathcal{A}.$$

Note, in particular, that each of $\mathcal{A}$ and $\mathcal{B}$ determines the other.

3. Triangulated subcategories associated to a phylum

Given a phylum $\mathbf{f}$, we define a full subcategory of $D^-(\mathbf{A}_W)$ as follows:

$$D^-(\mathbf{A}_W)_{\preceq \mathbf{f}} = \left\{ X \in D^-(\mathbf{A}_W) \;\middle|\; \begin{array}{c} X \text{ is isomorphic to a bounded-above} \\ \text{complex } M^\bullet \text{ where each } M^i \text{ is a direct} \\ \text{sum of various } P_\chi\langle n\rangle \text{ with } [\chi] \preceq \mathbf{f} \end{array} \right\}.$$

We will also consider the "strict" version $D^-(\mathbf{A}_W)_{\prec \mathbf{f}}$, as well as the analogous subcategories of $D^{\mathrm{b}}(\mathbf{A}_W)$ and $D^+(\mathbf{A}_W)$. It follows from (2.2) that

$$\mathbb{D}(D^-(\mathbf{A}_W)_{\preceq \mathbf{f}}) = D^+(\mathbf{A}_W)_{\preceq \mathbf{f}} \qquad \text{and} \qquad \mathbb{D}(D^{\mathrm{b}}(\mathbf{A}_W)_{\preceq \mathbf{f}}) = D^{\mathrm{b}}(\mathbf{A}_W)_{\preceq \mathbf{f}}. \tag{3.1}$$

In addition, we have

$$D^{\mathrm{b}}(\mathbf{A}_W)_{\preceq \mathbf{f}} = D^-(\mathbf{A}_W)_{\preceq \mathbf{f}} \cap D^{\mathrm{b}}(\mathbf{A}_W) = D^+(\mathbf{A}_W)_{\preceq \mathbf{f}} \cap D^{\mathrm{b}}(\mathbf{A}_W).$$

The first of these holds by a routine homological-algebra argument for bounded-above complexes of projectives over a ring with finite global dimension. The second equality follows from the first using (3.1).

In this section, we first construct a collection of objects in $D^-(\mathbf{A}_W)$ and $D^+(\mathbf{A}_W)$ with various Hom-vanishing properties related to the categories defined above. Then, under the additional assumption that these objects lie in $D^{\mathrm{b}}(\mathbf{A}_W)$, we prove structural results for that category in the spirit of Lemma 2.3.

3.1. Construction of ∇_χ and Δ_χ. We begin with the following result.

PROPOSITION 3.1. *For each $\chi \in \mathrm{Irr}(W)$, there is an object $\nabla_\chi \in D^-(\mathbf{A}_W)$ together with a morphism $s : P_\chi \to \nabla_\chi$ with the following properties:*

(1) *The cone of s lies in $D^-(\mathbf{A}_W)_{\prec [\chi]}$.*
(2) *For $M \in D^-(\mathbf{A}_W)_{\prec [\chi]}$ or $D^+(\mathbf{A}_W)_{\prec [\chi]}$, we have $\mathrm{Hom}(M, \nabla_\chi) = 0$.*

Moreover, the pair (∇_χ, s) is unique up to unique isomorphism.

PROOF. Given a module $M \in \mathbf{A}_W$-gmod, let $M_{\prec[\chi]}$ be the $\mathbf{A}_W$-submodule generated by all the homogeneous W-stable subspaces that are isomorphic to some $L_\psi\langle m\rangle$ with $\psi \prec \chi$. Of course, $M_{\prec[\chi]}$ is actually generated by a finite number of such subspaces. Thus, there is a surjective map $M' \twoheadrightarrow M_{\prec[\chi]}$, where M' is a direct sum of finitely many objects of the form $P_\psi\langle n\rangle$ with $\psi \prec \chi$.

We now define a complex $(N^\bullet, d^\bullet)$ inductively as follows. Let $N^i = 0$ for $i > 0$, and let $N^0 = P_\chi$. Then, assuming that N^i and $d^i : N^i \to N^{i+1}$ have already been defined for $i > j$, let us apply the construction of the preceding paragraph to $M = \ker d^{j+1} \subset N^{j+1}$. Set $N^j = M'$, and then let $d^j : N^j \to N^{j+1}$ be the composition

$$N^j \twoheadrightarrow (\ker d^{j+1})_{\prec[\chi]} \hookrightarrow N^{j+1}.$$

Let $\nabla_\chi = (N^\bullet, d^\bullet)$. There is an obvious morphism $s : P_\chi \to \nabla_\chi$. Its cone is isomorphic to the complex obtained from $(N^\bullet, d^\bullet)$ by omitting N^0. By construction, the N^i for $i < 0$ are direct sums of $P_\psi\langle n\rangle$ with $\psi \prec \chi$, so it is clear that the cone of s lies in $D^-(\mathbf{A}_W)_{\prec[\chi]}$.

For $M \in D^-(\mathbf{A}_W)_{\prec[\chi]}$ given by a suitable bounded-above complex of projectives, it is a routine exercise in homological algebra to show that any map $M \to \nabla_\chi$ is null-homotopic. On the other hand, if $M \in D^+(\mathbf{A}_W)_{\prec[\chi]}$ is given by a bounded-below complex of projectives, let M' be the subcomplex obtained by omitting the terms in degrees ≤ 1, and form a distinguished triangle $M' \to M \to M'' \to$. Then M'' lies in $D^{\mathrm{b}}(\mathbf{A}_W)_{\prec[\chi]}$. It is clear that $\mathrm{Hom}(M', \nabla_\chi) = \mathrm{Hom}(M'[1], \nabla_\chi) = 0$, and thus $\mathrm{Hom}(M, \nabla_\chi) = \mathrm{Hom}(M'', \nabla_\chi) = 0$ as well.

Finally, suppose $s' : P_\chi \to \nabla'_\chi$ were another morphism with the same properties, and let C' be its cone. Since $\mathrm{Hom}(C[-1], \nabla'_\chi) = 0$, the map s' factors through s, and then the last assertion follows by a standard argument. □

REMARK 3.2. In the construction above, it is easy to see by induction that the complex $(N^\bullet, d^\bullet)$ representing ∇_χ can be chosen such that each nonzero N^j is generated in grades $\geq -j \geq 0$. It follows that ∇_χ has grades ≥ 0.

PROPOSITION 3.3. *For each* $\chi \in \mathrm{Irr}(W)$, *there is an object* $\Delta_\chi \in D^+(\mathbf{A}_W)$ *together with a morphism* $t : \Delta_\chi \to P_\chi$ *with the following properties:*

(1) *The cone of* t *lies in* $D^+(\mathbf{A}_W)_{\prec[\chi]}$.
(2) *For* $M \in D^-(\mathbf{A}_W)_{\prec[\chi]}$ *or* $D^+(\mathbf{A}_W)_{\prec[\chi]}$, *we have* $\mathrm{Hom}(\Delta_\chi, M) = 0$.

Moreover, the pair (Δ_χ, t) *is unique up to unique isomorphism.*

PROOF. Let $\Delta_\chi = \mathbb{D}(\nabla_{\bar\chi})$, and let $t = \mathbb{D}(s) : \Delta_\chi \to P_\chi$. It follows from (2.2), (3.1), and Proposition 3.1 that (Δ_χ, t) has the required properties. □

COROLLARY 3.4. (1) *If* $\chi \not\sim \psi$, *then* $\underline{\mathrm{Hom}}^\bullet(\Delta_\chi, \nabla_\psi) = 0$.
(2) *If* $i > 0$, *then* $\underline{\mathrm{Hom}}^i(\Delta_\chi, \nabla_\psi) = 0$ *for all* χ, ψ.

PROOF. The first assertion follows from Propositions 3.1(2) and 3.3(2). For the second, observe that by construction, ∇_ψ is isomorphic to a complex of projectives in nonpositive degrees, so Δ_ψ is isomorphic to a complex of projectives in nonnegative degrees. The result then follows by the remarks after (2.1). □

3.2. Admissible subcategories of $D^{\mathrm{b}}(\mathbf{A}_W)$. For the remainder of this section, we impose the additional assumption that all the Δ_χ and ∇_χ lie in $D^{\mathrm{b}}(\mathbf{A}_W)$.

With this assumption, it makes sense to consider the following full triangulated subcategories of $D^{\mathrm{b}}(\mathbf{A}_W)$:

$$D^{\mathrm{b}}(\mathbf{A}_W)_{\mathbf{f}} = \text{the triangulated subcategory generated by the } \nabla_\chi\langle n\rangle \text{ with } \chi \in \mathbf{f},$$
$$D^{\mathrm{b}}(\mathbf{A}_W)^{\mathbf{f}} = \text{the triangulated subcategory generated by the } \Delta_\chi\langle n\rangle \text{ with } \chi \in \mathbf{f}.$$

We will see below that these two categories are equivalent. It often happens that the ∇_χ are easier to work with explicitly than the Δ_χ, so this equivalence will be useful for transferring facts about the former to the setting of the latter.

PROPOSITION 3.5. *For each phylum* $\mathbf{f}$, $D^{\mathrm{b}}(\mathbf{A}_W)_{\preceq\mathbf{f}}$ *is generated as a triangulated category by the* $\nabla_\chi\langle n\rangle$ *(resp. the* $\Delta_\chi\langle n\rangle$*) with* $[\chi] \preceq \mathbf{f}$.

PROOF. This follows by induction on $\mathbf{f}$ with respect to the order on the set of phyla, using the distinguished triangle $P_\chi \to \nabla_\chi \to C \to$ with $C \in D^{\mathrm{b}}(\mathbf{A}_W)_{\prec\mathbf{f}}$. □

In the case of the ∇_χ, this statement can be refined a bit. Recall from Remark 3.2 that ∇_χ has grades ≥ 0. It follows that in the distinguished triangle $P_\chi \to \nabla_\chi \to C \to$, the object C has grades ≥ 0. By keeping track of grades in the induction, one can see that each P_ψ is contained in the triangulated category generated by the $\nabla_\chi\langle k\rangle$ with $k \geq 0$. We have just shown that part (2) in the corollary below implies part (3). (Note, in contrast, that the Δ_χ do not, in general, have grades ≥ 0.)

COROLLARY 3.6. *The following conditions on an object* $M \in D^{\mathrm{b}}(\mathbf{A}_W)$ *are equivalent:*

(1) *M has grades $\geq n$.*
(2) *M is isomorphic to a complex of projective $\mathbf{A}_W$-modules each term of which has grades $\geq n$.*
(3) *M lies in the triangulated subcategory generated by the $\nabla_\chi\langle k\rangle$ with $k \geq n$.*

PROOF. We saw above that (2) implies (3). It is a routine exercise to see that (1) implies (2), and Remark 3.2 tells us that (3) implies (1). □

COROLLARY 3.7. *Each of the two pairs of categories* $(D^{\mathrm{b}}(\mathbf{A}_W)_{\mathbf{f}}, D^{\mathrm{b}}(\mathbf{A}_W)_{\prec\mathbf{f}})$ *and* $(D^{\mathrm{b}}(\mathbf{A}_W)_{\prec\mathbf{f}}, D^{\mathrm{b}}(\mathbf{A}_W)^{\mathbf{f}})$ *is an admissible pair in* $D^{\mathrm{b}}(\mathbf{A}_W)_{\preceq\mathbf{f}}$.

PROOF. This follows from Propositions 3.1(2), 3.3(2), and 3.5. □

The next two results are just restatements of parts (4)–(6) of Lemma 2.3.

PROPOSITION 3.8. *Let* $\mathbf{f}$ *be a phylum, and let* $M \in D^{\mathrm{b}}(\mathbf{A}_W)$. *The following three conditions are equivalent:*

(1) $M \in D^{\mathrm{b}}(\mathbf{A}_W)_{\prec\mathbf{f}}$.
(2) $\underline{\mathrm{Hom}}^\bullet(M, \nabla_\chi) = 0$ *for all* χ *with* $[\chi] \succeq \mathbf{f}$.
(3) $\underline{\mathrm{Hom}}^\bullet(\Delta_\chi, M) = 0$ *for all* χ *with* $[\chi] \succeq \mathbf{f}$. □

LEMMA 3.9. *The inclusion functors* $D^{\mathrm{b}}(\mathbf{A}_W)_{\mathbf{f}} \to D^{\mathrm{b}}(\mathbf{A}_W)$ *and* $D^{\mathrm{b}}(\mathbf{A}_W)^{\mathbf{f}} \to D^{\mathrm{b}}(\mathbf{A}_W)$ *induce equivalences of categories*

$$D^{\mathrm{b}}(\mathbf{A}_W)_{\mathbf{f}} \xrightarrow{\sim} D^{\mathrm{b}}(\mathbf{A}_W)_{\preceq\mathbf{f}}/D^{\mathrm{b}}(\mathbf{A}_W)_{\prec\mathbf{f}} \xleftarrow{\sim} D^{\mathrm{b}}(\mathbf{A}_W)^{\mathbf{f}}. \quad \square$$

Let us denote the composition of these two equivalences by

$$T_{\mathbf{f}} : D^{\mathrm{b}}(\mathbf{A}_W)_{\mathbf{f}} \xrightarrow{\sim} D^{\mathrm{b}}(\mathbf{A}_W)^{\mathbf{f}}. \tag{3.2}$$