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ROBUST CONTROL DESIGN 2000

A Proceedings volume from the 3rd IFAC Symposium
Prague, Czech Republic, 21 - 23 June 2000

Edited by

V. KUČERA and M. ŠEBEK

Volume 1

PERGAMON

ROBUST CONTROL DESIGN 2000
(ROCOND 2000)

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Foreword

The 3rd IFAC ROCOND Symposium on Robust Control Design took place in Hotel Renaissance, Prague, Czech Republic during June 21 – 23, 2000.

The ROCOND symposia have developed from an IFAC conference on Control System Design, which was held in Zűrich, Switzerland in 1991. The 1st IFAC ROCOND Symposium was organized in Rio de Janeiro, Brazil in 1994. The series then continued by the 2nd IFAC ROCOND Symposium in Budapest, Hungary in 1997.

Following the tradition, the aim of ROCOND 2000 was to bring together the robust control community to discuss the trends in the field and to present new methods and applications.

The technical program included 21 sessions on robust control and related topics in identification and signal processing. The methods presented in these sessions included linear matrix inequalities, polynomial techniques, sliding modes, optimal control, fuzzy and adaptive control. Attention was paid to linear as well as nonlinear systems.

The highlights of the technical program were two plenary lectures by world famous experts in the field: Robust Control and Filtering Design for Discrete-Time Systems, by J. Geromel (Universidade Estadual de Campinas, Brazil) and H2-optimization: Theory and Application to Robust Control Design, by H. Kwakernaak (Twente University, Netherlands).

Very well attended was the invited application session on Parameter-Space Tools for Robust Control, organized by J. Ackermann (German Aerospace Research Establishment, Oberpfaffenhofen). Part of the Symposium was an Europoly Workshop, which included 3 sessions of high quality papers on the theory and applications of polynomial design methods in control and signal processing.

It is my pleasure to congratulate U. Shaked (Tel Aviv University, Israel), the chairman of the International Program Committee and M. Šebek (Czech Technical University in Prague, Czech Republic), the chairman of the National Organizing Committee for their achievements in preparing and running the Symposium. I hope that the robust control community enjoyed this event and already look forward to the 4th IFAC ROCOND Symposium to be held in Milano, Italy in the year 2003.

Vladimír Kučera
Editor

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A POLYNOMIAL APPROACH TO ℓ_1 OPTIMAL CONTROL PROBLEMS

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Abstract: In this paper, the scalar multi-block ℓ_1-optimal control problem is considered. It is shown that it can be converted via polynomial equation techniques to an infinite dimensional linear programming (LP) problem. Finite dimensional sub/super approximations can be determined by considering two sequences of modified finite dimensional linear programming problems derived directly from the YJBK parameterization by exploiting the underlying algebraic structure. This approach induces the application of a consistent truncation strategy that leads to a redundancy-free constraint formulation and, as a consequence, to linear programming problems less affected by degeneracy. Further, more insight on the algebraic structure of the problem and on the achievement of exact rational solutions is provided, allowing the development of a simple and conceptually attractive theory. *Copyright ©2000 IFAC*

Keywords: ℓ_1 Optimal Control, Linear Programming, Polynomials equation, Discrete-time Systems.

Definitions and notations

$\mathcal{Z}(\cdot)$: The $\mathcal{Z}$-transform operator. Given a matrix $\hat{H} = \{\hat{H}(k)\}_{k=0}^{\infty}$ of causal real sequences: $\mathcal{Z}(\hat{H}) = \mathcal{H}(d) := \sum_{k=0}^{\infty} \mid \hat{H}(k) \mid d^k$, in the complex variable d.

$\ell_{1,p\times m}$: The real normed linear space of all $p \times m$ matrices $\hat{H}$ of absolutely summable real causal sequences $\hat{H}_{ij} = \{\hat{H}_{ij}(k)\}_{k=0}^{\infty}$ with norm $\|\hat{H}\|_1 := \max_{i\in \underline{p}} \sum_{j=1}^{m} \sum_{k=0}^{\infty} \mid \hat{H}(k) \mid < \infty$, where $\underline{p} := \{1, 2, ..., p\}$.

$\mathcal{A}_{p\times m}$: The real normed linear space of all $p \times m$ matrices $\mathcal{H}(d)$ which are $\mathcal{Z}$-transform of some matrix sequence $\hat{H} \in \ell_{1,p\times m}$. Note that $\mathcal{A}$ is isomorphic to $\ell_{1,p\times m}$.

$\mathcal{R}\ell_{1,p\times m}$: The subspace of $\ell_{1,p\times m}$ of all real causal matrix sequences each of whose entries has a rational $\mathcal{Z}$-transform.

$\mathcal{R}A$: The subspace of $\mathcal{A}_{p\times m}$ consisting of elements each of whose entries is a stable stable real rational functions.

$\partial(X)$: Denotes the degree of the polynomial X.

In the sequel, with the *hat* we denote (matrix) sequences, whereas the same *unhatted* variable denotes its corresponding $\mathcal{Z}$-transform; the space of polynomials will be denoted by $\mathcal{R}[d]$ whereas rational transfer functions by $\mathcal{R}(d)$. With $\mathcal{R}^t[d]$ will be denoted all polynomial with degree lower than or equal to t.

1. INTRODUCTION

The discrete-time model matching ℓ_1-optimization problem (Dahleh and Pearson, 1987; Dahleh and Pearson, 1988) amounts to the minimization of the $\mathcal{A}$-norm of the closed-loop error transfer matrix

$$\mathcal{E}(d) = \mathcal{H}(d) - \mathcal{U}(d)\mathcal{Q}(d)\mathcal{V}(d) \qquad (1)$$

where $\mathcal{H}$, $\mathcal{U}$, and $\mathcal{V}$ are given stable rational matrices ($\in \mathcal{R}A$) of appropriate dimensions and $\mathcal{Q} \in \mathcal{A}$ is the free parameter. It is now well understood that one-block problems (MacDonald and Pearson, 1991; Staffans, 1991) admit rational solutions, that is there exists a minimizer $\mathcal{Q}_o \in \mathcal{RA}$ provided that $\mathcal{U}$ and $\mathcal{V}$ have not transmission zeros on the unit circle.

In bad-rank cases such a strong result of existence is not so far available and minimizers were shown to exist only in $\mathcal{A}$. In this case, finite dimensional approximations are of interest, along with an estimate of the gap existing between the corresponding sub-optima and the true optimum. Various strategies have been proposed to that purpose: Q-design (Boyd and Barratt, 1991), FMV-FME (Staffans, 1993), Delay Augmentation Method (Diaz-Bobillo and Dahleh, 1993), Semidefinite (Quadratic) programming methods (Elia and Dahleh, 1998). All of them, except the last method, consist of adding additional constraints to the original optimization problem in order to achieve sub-optimal finite dimensional solutions. Further, in order to have an estimate of the quality of the approximated solution, in (Staffans, 1993) was firstly introduced the idea of building a dual sequence of linear programming problems whose solutions form a not-decreasing sequence of super-optima converging from the below to the optimum. The latter was obtained by dropping some of the structural constraints existing between $\mathcal{E}$ and $\mathcal{Q}$ in (1). The combined use of such a sub/super-optimization scheme allows one to obtain sequences of lower and upper bounds both converging to the optimum. The Semidefinite programming approach (Elia and Dahleh, 1998) instead, embeds the original ℓ_1 problem into two finite dimensional $\mathcal{H}_2$ quadratic programming problems (sub- and super-optimal) and uses the LMI paradigm to numerically approach the problem.

The main goal of this paper is to present a different characterization of the closed-loop error transfer function (1). This is done by resorting the polynomial equation approach of Kučera (Kucěra, 1979). This allows a more direct achievement of unconstrained suboptimal and superoptimal linear programming problems that are free of most of the above defects. The key idea consists of parameterizing both the closed-loop error $\mathcal{E}$ and the free parameter $\mathcal{Q}$ in terms of a polynomial matrix, that really represents the available degrees of freedom existing as long as the closed-loop stability and feedback structural constraints underlying (1) have been satisfied. As a consequence, the original optimization problem can be expressed in terms of this new polynomial matrix, resulting in an *unconstrained* linear programming problem.

Such a scheme was considered in (Casavola, 1996) for scalar mixed-sensitivity problems. Here the 2×2 multi-block case is considered. The case treated exhausts the ideas involved in the present polynomial approach and extends directly to the general MIMO case, that will be presented elsewhere.

2. PROBLEM FORMULATION

Consider the following system

$$\mathcal{G}(d) = \begin{bmatrix} \mathcal{G}_{11}(d) & \mathcal{G}_{12}(d) \\ \mathcal{G}_{21}(d) & \mathcal{G}_{22}(d) \end{bmatrix}, \quad \mathcal{G}_{ij}(d) \in \mathcal{R}(d), \quad (2)$$

the YJBK parameterization of all admissible closed-loop error maps can be characterized by a matrix transfer function $\mathcal{H}$ and polynomial matrices U and V as

$$\mathcal{E} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{bmatrix} - \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \mathcal{Q} \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \quad (3)$$

for some $\mathcal{Q} \in \mathcal{A}$. In (3), we can assume w.l.o.g. that $\mathcal{H}_{ij}(d) \in \mathcal{RA}$, $U_i \in \mathcal{R}[d]$ and $V_i \in \mathcal{R}[d]$, $i = 1, 2$. Let $\mathcal{H}_{ij} = N_{ij}/D$, $\mathcal{Q} = P/(DS)$ be defined as polynomial ratios for some given polynomials N_{ij} and D, with D strictly-Schur, viz. all of their possible zeros are outside of the unit disk, and S and P free with S strictly-Schur and $P \in \mathcal{A}$ of possibly infinite degree. Further, let $\bar{U}(d) := \gcd(U_1, U_2)$ and $\bar{V}(d) := \gcd(V_1, V_2)$ denote the respective greatest common polynomial divisors so that $U_i = \bar{U}\hat{U}_i$ and $V_i = \bar{V}\hat{V}_i$. Throughout the paper we assume that

(A.1) $\{$ • $\bar{U}(d)$, $\bar{V}(d)$ have no roots over the unit circle.

The problem we want to solve is the following ℓ_1 four-block model-matching problem

$$\mu_{opt} := \inf_{\mathcal{Q} \in \mathcal{A}} \|\mathcal{E}\|_{\mathcal{A}} \quad (4)$$

2.1 *Structural conditions*

In order to solve the problem we first characterize the class of all admissible closed-loop maps $\mathcal{E} = (\mathcal{E}_{ij})$, $i, j = 1, 2$, viz. compatible with the feedback structure (3). To this end, for the sake of clarity, we consider first the case in which the terms $\mathcal{H}_{i1}\hat{V}_2 - \mathcal{H}_{i2}\hat{V}_1$, $i = 1, 2$, and $\mathcal{H}_{1j}\hat{U}_2 - \mathcal{H}_{2j}\hat{U}_1$, $j = 1, 2$ are polynomials. Specifically, denote as T_i, $i = 1, \ldots, 4$

$$\mathcal{H} \begin{bmatrix} \hat{V}_2 \\ -\hat{V}_1 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} \hat{U}_2 & -\hat{U}_1 \end{bmatrix} \mathcal{H} = \begin{bmatrix} T_3 & T_4 \end{bmatrix}. \quad (6)$$

Note that the following identity holds true

$$T_1\hat{U}_2 + T_4\hat{V}_1 = T_2\hat{U}_1 + T_3\hat{V}_2. \quad (7)$$

Under the above assumption we have.

Lemma 1 - Let T_i in (5)-(6) be polynomials. Then, the all and the only $\mathcal{E}_{ij}(d)$ that jointly satisfy (3) can be parameterized in terms of a possibly infinite degree free polynomial $X(d) \in \mathcal{A}$ as

$$\mathcal{E}_{ij} = E_{ij}^0 + \hat{U}_i \hat{V}_j X \quad (8)$$

In (8), $E_{ij}^0 \in \mathcal{R}[d]$ are particular solutions of the following pair of uncoupled polynomial Diophantine equations

$$E_{11}\hat{U}_2\hat{V}_2 - E_{22}\hat{U}_1\hat{V}_1 = T_2\hat{U}_1 + T_3\hat{V}_2\,, \quad (9)$$
$$E_{21}\hat{U}_1\hat{V}_2 - E_{12}\hat{U}_2\hat{V}_1 = T_1\hat{U}_2 - T_3\hat{V}_2\,. \quad (10)$$

Remark 1 - Notice that the above parameterization characterizes the class of all admissible responses with finite support (deadbeat responses) whenever $X(d)$ is restricted to be a polynomial of finite degree. A convenient choice in order to avoid degree inflation in the solution is to select for E_{ij}^0 as the minimal degree solutions of (9) and (10). However, it is worth pointing out that the minimal degree solutions of (9) and (10) with respect to the first or the second of their arguments may differ in general. This does not happen, e.g. for (9), when $\partial(\hat{U}_1\hat{V}_2) + \partial(\hat{V}_1\hat{U}_2) > \partial(T_1\hat{U}_2 - T_3\hat{V}_2)$, where $\partial(T)$ denotes the degree of the polynomial T. A similar condition holds for (10) □

2.2 *Stability conditions*

The assumptions (5)-(6) imply also that

$$\left(N_{i1} - DE_{i1}^0\right)\hat{V}_2 = \left(N_{i2} - DE_{i2}^0\right)\hat{V}_1,, \quad (11)$$
$$\left(N_{1j} - DE_{1j}^0\right)\hat{U}_2 = \left(N_{2j} - DE_{2j}^0\right)\hat{U}_1, \quad (12)$$

where $i = 1, 2, j = 1, 2$. From the above relations it follows that there exists a single polynomial $W(d)$ such that

$$N_{ij} - DE_{ij}^0 = \hat{U}_i\hat{V}_j W, \quad i,j = \{1,2\}\,, \quad (13)$$

which implies that E_{ij}^0 interpolate $\frac{N_{ij}}{D}$ on the roots of $\hat{U}_i\hat{V}_j$. This consideration is not sufficient for ensuring closed-loop $\mathcal{A}$-stability. In fact, rewrite the error sequences $\mathcal{E}_{ij}$ as

$$E_{ij}^0 + \hat{U}_i\hat{V}_j X = \frac{N_{ij}}{D} - \hat{U}_i\hat{V}_j\bar{U}\bar{V}\frac{P}{DS}\,, \quad (14)$$

where $Q = \dfrac{P}{DS}$ and P and S polynomials to be determined. From (14) one obtains

$$DQ = \frac{P}{S} = \frac{W - DX}{\bar{U}\bar{V}}\,, \quad (15)$$

where the last equality follows from (13). Then, in order to ensure $\mathcal{A}$-stability is necessary and sufficient that

$$P = \bar{U}^-\bar{V}^-P_1\,, \quad (16)$$

for a possibly infinite degree polynomial $P_1 \in \mathcal{A}$. In (16), we have factorized $\bar{U} = \bar{U}^-\bar{U}^+$, $\bar{U}^+$ strictly-Schur and $\bar{U}^-$ monic anti-Schur and the same for $\bar{V} = \bar{V}^-\bar{V}^+$. Now, all polynomials P_1 and X that satisfy (15) must satisfy equivalently the following Diophantine equation

$$\bar{U}^-\bar{V}^-P_1 + DX = W\,. \quad (17)$$

The above equation is always solvable because $(\bar{U}^-\bar{V}^-, D)$ are coprime. Let (X_0, P_{10}) the minimum degree solution of (17) w.r.t. X, that is $\partial(X) < \partial(\bar{U}^-\bar{V}^-)$. Then, the general solution has the following expression

$$\begin{cases} X \;=\; X_0 - \bar{U}^-\bar{V}^-T\,, \\ P_1 \;=\; P_{10} + DT\,. \end{cases} \quad (18)$$

with $T(d)$ is a free polynomial. As a consequence, all admissible and $\mathcal{A}$-stable closed-loop error maps $\mathcal{E}_{ij}$ can be parameterized in terms of a possibly infinite degree free polynomial $T \in \mathcal{A}$ as

$$\mathcal{E}_{ij} = E_{ij}^0 + \hat{U}_i\hat{V}_j X_0 + \hat{U}_i\hat{V}_j\left(\bar{U}^-\bar{V}^-T\right)\,. \quad (19)$$

If each E_{ij}^0 is the minimal degree solution, we know that $\partial(E_{ij}^0) < \partial(\hat{U}_i\hat{V}_j)$. Moreover $\partial(X_0) < \partial(\bar{U}^-\bar{V}^-)$ and $\partial(E_{ij}^0 + \hat{U}_i\hat{V}_j X_0) < \partial(\hat{U}_i\hat{V}_j\bar{U}^-\bar{V}^-)$. Then, if T is a polynomial of degree t we have that

$$\partial(\mathcal{E}_{ij}) < \partial(\hat{U}_i\hat{V}_j\,\bar{U}^-\bar{V}^-) + \partial(T) < m_0 + t\,. (20)$$

where $m_0 := \max_{i,j}\partial(\hat{U}_i\hat{V}_j\,\bar{U}^-\bar{V}^-)$.

The parameterization (19) hinges upon the limitative assumptions (5)-(6). When they don't hold true, one can adopt a truncating strategy. Several equivalent alternatives are possible. A simple idea, exemplified for (5), consists of finding a polynomial pair $(T_1^{(N)}, \tilde{T}_1^{(N)})$, with $T_1^{(N)}$ of degree lower or equal to N, such that

$$\frac{N_{11}}{D}\hat{V}_2 - \frac{N_{12}}{D}\hat{V}_1 = T_1^{(N)} + d^{N+1}\frac{\tilde{T}_1^{(N)}}{D}\,, \quad (21)$$

and similarly for (6). Such a polynomial pair is unique for any N and can be computed as the minimal degree solution w.r.t. $T_i^{(N)}$ of the following set of Diophantine equations

$$DT_1^{(N)} + d^{N+1}\tilde{T}_1^{(N)} = N_{11}\hat{V}_2 - N_{12}\hat{V}_1\,, \quad (22)$$
$$DT_2^{(N)} + d^{N+1}\tilde{T}_2^{(N)} = N_{21}\hat{V}_2 - N_{22}\hat{V}_1\,, \quad (23)$$
$$DT_3^{(N)} + d^{N+1}\tilde{T}_3^{(N)} = N_{11}\hat{U}_2 - N_{21}\hat{U}_1\,, \quad (24)$$
$$DT_4^{(N)} + d^{N+1}\tilde{T}_4^{(N)} = N_{12}\hat{U}_2 - N_{22}\hat{U}_1\,. \quad (25)$$

Notice that (22)-(25) are always solvable with $\partial(T_i^{(N)}) < N+1$ because (D, d^{N+1}) are coprime for all N. Then, the parameterization (8) of Lemma 1 becomes

$$\mathcal{E}_{ij} = E_{ij}^{(N)} + \hat{U}_i\hat{V}_j X + \frac{d^{N+1}}{D}\tilde{E}_{ij}^{(N)}, \quad (26)$$

where $E_{ij}^{(N)}$ and $\tilde{E}_{ij}^{(N)}$ are the minimal degree solutions (see Remark 1 for details) of the Diophantine equations (9) and (10) with T_i replaced by $T_i^{(N)}$ and, respectively, $\tilde{T}_i^{(N)}$, $i = 1,2,3,4$.

Next, by observing that eqs. (11)-(12) modify coherently for all integer N $i = 1,2$, $j = 1,2$, and similarly for one finally concludes that there exists a single polynomial $\bar{W}$, independent of N, such that

$$\frac{-DE_{ij}^N - d^{N+1}\tilde{E}_{ij}^{(N)} + N_{ij}}{\hat{U}_i\hat{V}_j} = \bar{W}, \quad (27)$$

Again, the stability issue is resolved by requiring that

$$DQ = \frac{P}{S} = \frac{\bar{W} - DX}{\bar{U}\bar{V}}, \quad (28)$$

is $\mathcal{A}$-stable, viz. by requiring that $P = \bar{U}^-\bar{V}^-P_1$, with $P_1 \in \mathcal{A}$ solution of the following Diophantine equation

$$\bar{U}^-\bar{V}^-P_1 + DX = \bar{W}. \quad (29)$$

Specifically, let (P_{10}, X_0) be the minimal degree solution w.r.t. X of (29), viz. $\partial(X_0) < \partial(\bar{U}^-\bar{V}^-)$. Then, the general solution of (29) is given by $X = X_0 - \bar{U}^-\bar{V}^-T$ and $P_1 = P_{10} + DT$ in the possibly infinite degree free polynomial $T \in \mathcal{A}$. Finally, for any integer N, the parameterization of all admissible and $\mathcal{A}$ stable closed-loop error maps $\mathcal{E}_{ij}$ (19) modify as

$$E_{ij}^{(N)} + \hat{U}_i\hat{V}_j X_0 - \hat{U}_i\hat{V}_j\bar{U}^-\bar{V}^-T + \frac{d^{N+1}}{D}\tilde{E}_{ij}^{(N)}. \quad (30)$$

3. SUBOPTIMIZATION

The parameterizations (19) and (30) allow one to directly construct suboptimization schemes by imposing that the closed-loop error maps are polynomials. In fact, the above conditions impose additional constraints to (4) and the corresponding solutions are of course sub-optimal. Then, any arbitrarily tight approximating solution to (4) can be obtained by solving the following finite-dimensional linear programming problem (**SUP-OPT**$_t$) for a sufficient large value for $t := \partial(T)$:

$$\min_{T \in \mathcal{R}^t[d]} \left\| E^{(N)} + \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} (X_0 + \bar{U}^-\bar{V}^-T) \begin{bmatrix} \hat{V}_1 & \hat{V}_2 \end{bmatrix} \right\|_{\mathcal{A}},$$

whose value is $\bar{\mu}_t$. A convenient choice for N, in order to have all coefficients of $E_{ij}^{(N)} + \hat{U}_i\hat{V}_jX_0$ influenced by T, is

$$N = N(t) = \min_{i,j=1,2} \partial(\hat{U}_i\hat{V}_j\bar{U}^-\bar{V}^-) + \partial(T). \quad (31)$$

Then, by denoting with

$$\tilde{\mu}_t := \left\| D^{-1}d^{N+1}E^{(N)} \right\|_{\mathcal{A}}, \quad (32)$$

the part of the cost due to truncated amounts a link with the OPT problem is established by the following Lemma.

Lemma 2 - Let $(A.1)$ be fulfilled and $T^{(t)}$ denote a solution of **SUB-OPT**$_t$. Then, the sequence $\bar{\mu}_t$ is non-increasing and

$$\bar{\mu}_t + \tilde{\mu}_t \geq \bar{\mu}_{t+1} + \tilde{\mu}_{t+1} \geq \mu_{opt}, \; \forall\, t \geq 0$$
$$\lim_{t\to\infty} \bar{\mu}_t = \mu_{opt}, \quad \lim_{t\to\infty} \tilde{\mu}_t = 0$$

Further, the sequence of solutions $T^{(t)}$ admits a subsequence $T^{(t_s)}$ that converges in the $\mathcal{A}$-norm (component-wise) to an optimal solution of the OPT problem as $t \to \infty$. If such a solution is unique, the whole sequence converges to it.

4. SUPEROPTIMIZATION

In order to derive linear programming problems whose solutions provide a sequence of lower-bounds to μ_{opt}, it is necessary to rule out some constraints from (4). Because one cannot eliminate constraints related to stability, the only possibility is to relax some structural constraints. This can be done easily by considering four free polynomials $(T + d^{t+1}T_{ij})$ instead of a single T in (19). It is evident that this choice remove some structural conditions depending on the degrees of T and T_{ij}. Once such substitutions have taken place, (19) becomes

$$\bar{\mathcal{E}}_{ij} = E_{ij}^0 + \hat{U}_i\hat{V}_jX_0 - U_iV_j(T + d^{t+1}T_{11}), \quad (33)$$

with $\bar{\mathcal{E}}_{ij}$ any longer admissible for OPT. A further simplification can be accomplished by considering that the closure of the space all polynomials $U(d)T(d) \in \mathcal{RA}$ (of possibly infinite degree) generated by an arbitrary polynomial $T(d) \in \mathcal{RA}$, with $U(d)$ polynomial, is given by $U(d)^*T(d)$, where $U(d)^*$ denotes the anti-Schur factor of $U(d)$ with all repeated zeros on the unit circle replaced by simple zeros (Vidyasagar, 1991). This means that $U(d)^*T(d) \in \mathcal{RA}$ if and only if $T(d) \in \mathcal{RA}$. In particular, $U(d)^*T(d)$ polynomial if and only if $T(d)$ is a polynomial. As a consequence, one can usefully consider the following substitutions in (33)

$$\bar{\mathcal{E}}_{ij} = E_{ij}^0 + \hat{U}_i\hat{V}_jX_0 + U_iV_jT + d^{t+1}U_i^*V_j^*T_{ij}, \quad (34)$$