SCHAUM'S OUTLINE OF

PRINCIPLES AND PROBLEMS

of

PLANE GEOMETRY

with Coordinate Geometry

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BY

BARNETT RICH, Ph.D.

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Preface

The central purpose of this book is to provide maximum help for the student and maximum service for the teacher.

PROVIDING HELP FOR THE STUDENT:

This book has been designed to improve the learning of geometry far beyond that of the typical and traditional book in the subject. Students will find this text useful for these reasons:

(1) ***Learning Each Rule, Formula and Principle***

Each rule, formula and principle is stated in simple language, is made to stand out in distinctive type, is kept together with those related to it, and is clearly illustrated by examples.

(2) ***Learning Each Set of Solved Problems***

Each set of solved problems is used to clarify and apply the more important rules and principles. The character of each set is indicated by a title.

(3) ***Learning Each Set of Supplementary Problems***

Each set of supplementary problems provides further application of rules and principles. A guide number for each set refers a student to the set of related solved problems. There are more than 2000 additional related supplementary problems. Answers for the supplementary problems have been placed in the back of the book.

(4) ***Integrating the Learning of Plane Geometry***

In accordance with the syllabus of the State of New York, the book integrates plane geometry with arithmetic, algebra, numerical trigonometry, coordinate geometry and simple logic. To carry out this integration:

(*a*) A separate chapter is devoted to coordinate geometry.

(*b*) A separate chapter includes the complete proofs of all the required theorems together with the plan for each.

(*c*) A separate chapter explains fully 27 basic geometric constructions, including all the required constructions. Underlying geometric principles are provided for the constructions, as needed.

(*d*) Two separate chapters on methods of proof and improvement of reasoning present the simple and basic ideas of formal logic suitable for students at this stage.

(*e*) Throughout the book, algebra is emphasized as the major means of solving geometric problems through algebraic symbolism, algebraic equations and algebraic proof.

(5) ***Learning Geometry Through Self-study***

The method of presentation in the book makes it ideal as a means of self-study. For the able student, this book will enable him to accomplish the work of the standard course of study in much less time. For the less able, the presentation of numerous illustrations and solutions provides the help needed to remedy weaknesses and overcome difficulties and in this way keep up with the class and at the same time gain a measure of confidence and security.

(6) *Extending Plane Geometry into Solid Geometry*

A separate chapter is devoted to the extension of two-dimensional plane geometry into three-dimensional solid geometry. It is especially important in this day and age that the student understand how the basic ideas of space are outgrowths of principles learned in plane geometry.

PROVIDING SERVICE FOR THE TEACHER:

Teachers of geometry will find this text useful for these reasons:

(1) *Teaching Each Chapter*

Each chapter has a central unifying theme. Each chapter is divided into two to ten major subdivisions which support its central theme. In turn, these chapter subdivisions are arranged in graded sequence for greater teaching effectiveness.

(2) *Teaching Each Chapter Subdivision*

Each of the chapter subdivisions contains the problems and materials needed for a complete lesson developing the related principles.

(3) *Making Teaching More Effective Through Solved Problems*

Through proper use of the solved problems, students gain greater understanding of the way in which principles are applied in varied situations. By solving problems, mathematics is learned as it should be learned — by doing mathematics. To ensure effective learning, solutions should be reproduced on paper. Students should seek the why as well as the how of each step. Once a student sees how a principle is applied to a solved problem, he is then ready to extend the principle to a related supplementary problem. Geometry is not learned through the reading of a textbook and the memorizing of a set of formulas. Until an adequate variety of suitable problems have been solved, a student will gain little more than a vague impression of plane geometry.

(4) *Making Teaching More Effective Through Problem Assignment*

The preparation of homework assignments and class assignments of problems is facilitated because the supplementary problems in this book are related to the sets of solved problems. Greatest attention should be given to the underlying principle and the major steps in the solution of the solved problems. After this, the student can reproduce the solved problems and then proceed to do those supplementary problems which are related to the solved ones.

OTHERS WHO WILL FIND THIS TEXT ADVANTAGEOUS:

This book can be used profitably by others beside students and teachers. In this group we include: (1) the parents of geometry students who wish to help their children through the use of the book's self-study materials, or who may wish to refresh their own memory of geometry in order to properly help their children; (2) the supervisor who wishes to provide enrichment materials in geometry, or who seeks to improve teaching effectiveness in geometry; (3) the person who seeks to review geometry or to learn it through independent self-study.

BARNETT RICH

Brooklyn Technical High School
April, 1963

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Chapter 1

Lines, Angles, and Triangles

1. Historical Background of Geometry

The word *geometry* is derived from the Greek words *geos* meaning *earth* and *metron* meaning *measure.* The ancient Egyptians, Chinese, Babylonians, Romans and Greeks used geometry for surveying, navigation, astronomy and other practical occupations.

The Greeks sought to systematize the geometric facts that had been discovered by establishing logical reasons for them and relationships among them. The work of such men as Thales (600 B.C.), Pythagoras (540 B.C.), Plato (390 B.C.) and Aristotle (350 B.C.) in systematizing geometric facts and principles culminated in the geometry text *Elements,* written about 325 B.C. by Euclid. This most remarkable text has been in use for over 2000 years.

2. Undefined Terms of Geometry: Point, Line and Surface

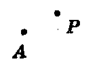
A. Point, Line, and Surface are Undefined Terms

These terms begin the process of definition and underlie the definitions of all other geometric terms. However, meanings can be given to these undefined terms by means of descriptions. These descriptions which follow are not to be thought of as definitions.

B. Point

A point has position only. It has no length, width, or thickness.

A point is represented by a dot. Keep in mind however that the dot represents a point but is not a point, just as a dot on a map may represent a locality but is not the locality. A dot, unlike a point, has size.

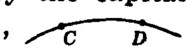
A point is designated by a capital letter next to the dot, thus: 

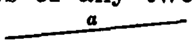
C. Line

A line has length but has no width or thickness.

A line may be represented by the path of a piece of chalk on the blackboard or by a stretched rubber band.

A line is designated by the capital letters of any two of its points or by a small letter, thus: 

, 

or 

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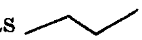
A line may be straight, curved, or a combination of these. To understand how lines differ, think of a line as being generated by a moving point.

A *straight line,* such as , is generated by a point moving in the same direction.

A *curved line,* such as , is generated by a point moving in a continuously changing direction.

A *broken line,* such as , is a combination of straight lines.

A straight line is unlimited in extent. It may be extended in either direction indefinitely.

A straight line is the shortest line between two points. Two straight lines intersect in a point.

D. Surface

A surface has length and width but no thickness. It may be represented by a blackboard, a side of a box or the outside of a sphere; these are representations of a surface but are not surfaces.

A *plane surface* or a *plane* is a surface such that a straight line connecting any two of its points lies entirely in it. A plane is a flat surface and may be represented by the surface of a flat mirror or the top of a desk.

Plane Geometry is the geometry that deals with plane figures that may be drawn on a plane surface. Unless otherwise stated, *figure* shall mean plane figure.

2.1 ILLUSTRATING UNDEFINED TERMS

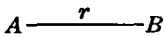
Point, line, and surface are undefined terms. State which of these terms is illustrated by each of the following:

(*a*) A light ray (*c*) A projection screen (*e*) A stretched thread
(*b*) The top of a desk (*d*) A ruler's edge (*f*) The tip of a pin

Ans. (*a*) line (*b*) surface (*c*) surface (*d*) line (*e*) line (*f*) point

3. Straight Line Segments

A straight line segment is the part of a straight line between two of its points. It is represented by the capital letters of these points or by a small letter. Thus AB or r represents the straight line segment between A and B.



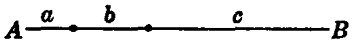
The expression *straight line segment* may be shortened to *line segment*, or *segment*, or even *line*, if the meaning is clear. Thus, line AB or AB means the straight line segment AB, unless otherwise stated.

Dividing a Line into Parts

If a line is divided into parts:

1. The whole line equals the sum of its parts.
2. The whole line is greater than any part.

   Thus if AB is divided into three parts a, b and c, then $AB = a + b + c$. Also, AB is greater than a; this may be written $AB > a$.

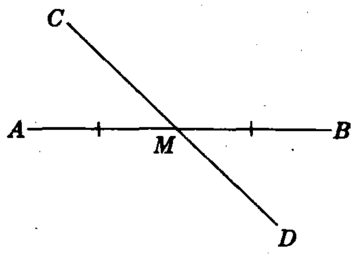


If a line is divided into two equal parts:

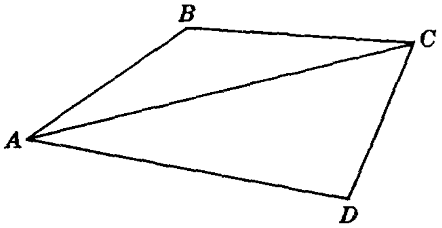
1. The point of division is the *midpoint of the line.*
2. A line that crosses at the midpoint is said *to bisect the given line.*

   Thus if $AM = MB$, then M is the midpoint of AB, and CD bisects AB.

   Equal line segments may be shown by crossing them with the same number of strokes. Note that AM and MB are crossed by a single stroke.



3.1 NAMING LINE SEGMENTS and POINTS



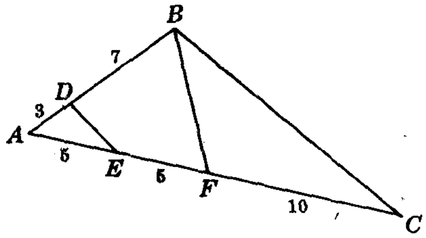
(a) Name each line segment shown.
(b) Name the line segments that intersect at A.
(c) What other line segment can be drawn?
(d) Name the point of intersection of CD and AD.
(e) Name the point of intersection of BC, AC and CD.

Solution:

(a) AB, BC, CD, AC, and AD. These segments may also be named by interchanging the letters; thus BA, CB, DC, CA, and DA are also correct.

(b) AB, AC, and AD (c) BD (d) D (e) C

3.2 FINDING LENGTHS and POINTS of LINE SEGMENTS



(a) State the lengths of AB, AC, and AF.
(b) Name two midpoints.
(c) Name two line bisectors.

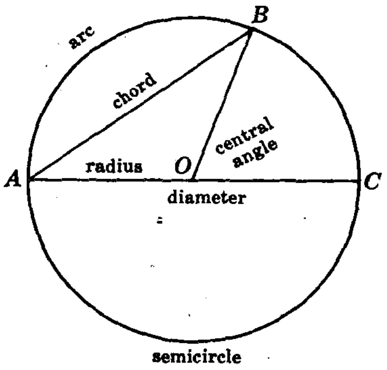
Solution:

(a) $AB = 3+7 = 10$, $AC = 5+5+10 = 20$, $AF = 5+5 = 10$.

(b) E is midpoint of AF, and F is midpoint of AC.

(c) DE is bisector of AF, and BF is bisector of AC.

4. Circles



A *circle* is a closed curve all of whose points are the same distance from the center. The symbol for circle is ⊙; for circles ⊚. Thus ⊙O stands for the circle whose center is O.

The *circumference* is the distance around the circle. It contains 360°.

A *radius* is a line joining the center to a point on the circumference. From the definition of a circle, it follows that radii of a circle are equal. Thus OA, OB and OC are radii of ⊙O and $OA = OB = OC$.

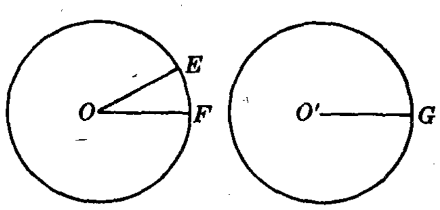
A *chord* is a line joining any two points on the circumference. Thus AB and AC are chords of ⊙O.

A *diameter* is a chord through the center of the circle; it is the longest chord. A diameter is twice a radius. Thus AC is a diameter of ⊙O.

An *arc* is a part of the circumference of a circle. The symbol for arc is ⌒. Thus $\widehat{AB}$ stands for arc AB. An arc of 1° is 1/360th of a circumference.

A *semicircle* is an arc equal to one-half of the circumference of a circle. A semicircle contains 180°. A diameter divides a circle into two semicircles. Thus diameter AC cuts ⊙O into two semicircles.

A *central angle* is an angle formed by two radii. Thus the angle between radii OB and OC is a central angle.

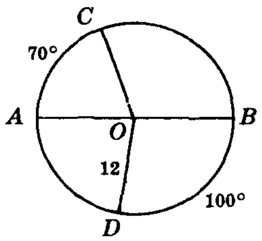


A *central angle of* 1° cuts off an arc of 1°. Thus if the central angle between OE and OF is 1°, then $\widehat{EF}$ is 1°.

Equal circles are circles having equal radii. Thus if $OE = O'G$, then circle O = circle O'.

4.1 FINDING LINES and ARCS in a CIRCLE

In circle O: (a) find OC and AB, (b) find the number of degrees in $\overset{\frown}{AD}$, (c) find the number of degrees in $\overset{\frown}{BC}$.



Solution:

(a) Radius OC = radius $OD = 12$. Diameter $AB = 24$.

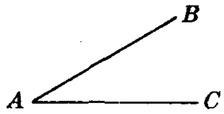
(b) Since semicircle $ADB = 180°$, $\overset{\frown}{AD} = 180° - 100° = 80°$.

(c) Since semicircle $ACB = 180°$, $\overset{\frown}{BC} = 180° - 70° = 110°$.

5. *Angles*

An *angle* is the figure formed by two straight lines meeting at a point. The lines are the sides of the angle while the point is its vertex. The symbol for angle is ∠. The plural is ∡.

Thus AB and AC are the sides of the angle, shown in Fig. (a), and A is its vertex.

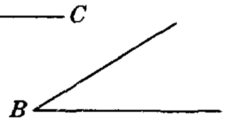


(a)

A. *Naming an Angle*

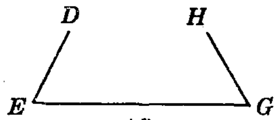
An angle may be named in any of the following ways:

1. The vertex letter if there is only one angle having this vertex, as $\angle B$, in Fig. (b).
2. A small letter or a number placed between the sides of the angle and near the vertex, as $\angle a$ or $\angle 1$, in Fig. (c).
3. Three capital letters with the vertex letter between two others on the sides of the angle. Referring to Fig. (d), $\angle E$ may be named $\angle DEG$ or $\angle GED$. $\angle G$ may be named $\angle EGH$ or $\angle HGE$.



(b)

(c)



(d)

B. *Measuring the Size of an Angle*

1. The size of an angle depends on the extent to which one side of the angle must be rotated or turned about the vertex until the turned side meets the other side.

   Thus the protractor in Fig. (e) shows that $\angle A$ is 60°. If AC were rotated about the vertex A until it met AB, the amount of turn would be 60°

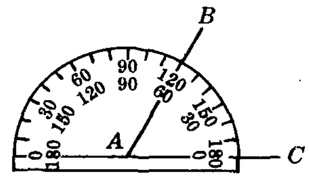
   In using a protractor, be sure that the vertex of the angle is at the center and that one side is along the 0°-180° diameter.

2. The size of an angle does *not* depend on the length of the sides of the angle.

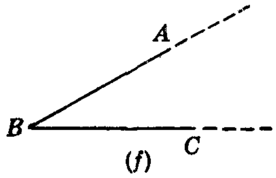
   Thus the size of $\angle B$, in Fig. (f), would not be changed if its sides AB and BC were made larger or smaller.

   No matter how large or small a clock is, the angle formed by its hands at 3 o'clock is 90°, as shown in Fig. (g).

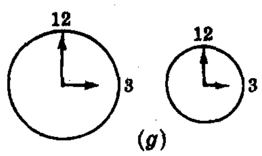
3. The Navy Compass, shown in Fig. (h), is read clockwise from 0° to 360°, beginning with North. A rotation from N to E is a quarter turn or 90°, that from NE to E is an eighth turn or 45°.



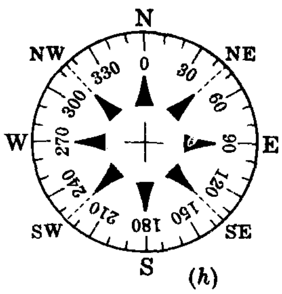
(e)



(f)



(g)

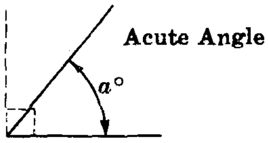


(h)

C. *Kinds of Angles:*

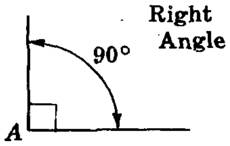
1. **Acute Angle** – *An acute angle is an angle that is less than 90°.*

   Thus, $a°$ is less than 90°; this is symbolized by $a° < 90°$.



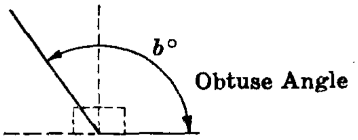
2. **Right Angle** – *A right angle is an angle that equals 90°.*

   Thus, rt. $\angle A = 90°$. The square corner denotes a right angle.



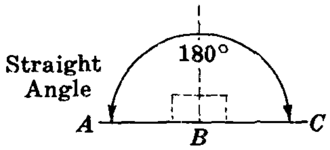
3. **Obtuse Angle** – *An obtuse angle is an angle that is more than 90° and less than 180°.*

   Thus, 90° is less than $b°$ and $b°$ is less than 180°; this is denoted by $90° < b° < 180°$.



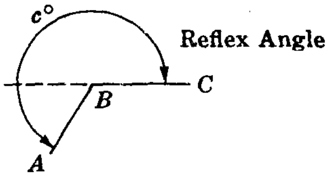
4. **Straight Angle** – *A straight angle is an angle that equals 180°.*

   Thus, st. $\angle B = 180°$. Note that the sides of a straight angle lie in the same straight line. However, do not confuse a straight angle with a straight line!



5. **Reflex Angle** – *A reflex angle is an angle that is more than 180° and less than 360°.*

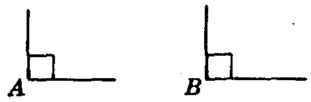
   Thus, 180° is less than $c°$ and $c°$ is less than 360°; this is symbolized by $180° < c° < 360°$.



D. *Additional Angle Facts:*

1. *Equal angles are angles that have the same number of degrees.*

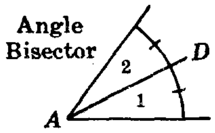
   Thus, rt. $\angle A$ = rt. $\angle B$ since each equals 90°.



2. *A line that bisects an angle divides it into two equal parts.*

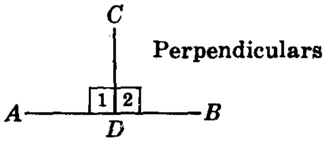
   Thus if AD bisects $\angle A$, then $\angle 1 = \angle 2$.

   Equal angles may be shown by crossing their arcs with the same number of strokes. Here the arcs of $\angle 1$ and $\angle 2$ are crossed by a single stroke.



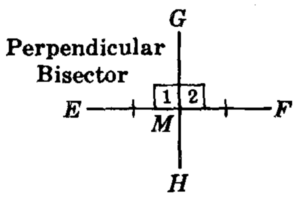
3. *Perpendiculars are lines that meet at right angles.*

   The symbol for perpendicular is $\perp$; for perpendiculars, $\perp$s. Thus if $CD \perp AB$, right angles 1 and 2 are formed.



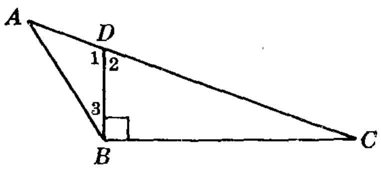
4. *A perpendicular bisector of a given line is both perpendicular to the line and bisects it.*

   Thus if GH is the $\perp$ bisector of EF, then $\angle 1$ and $\angle 2$ are right angles and M is the midpoint of EF.



5.1 NAMING an ANGLE

Name the following angles in the diagram: (*a*) Two obtuse angles, (*b*) a right angle, (*c*) a straight angle, (*d*) an acute angle at D, (*e*) an acute angle at B.

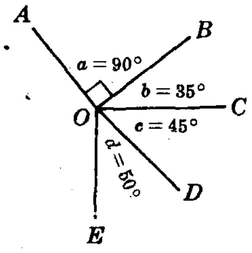


Solution:

(*a*) $\angle ABC$, and $\angle ADB$ or $\angle 1$. The angles may also be named by reversing the order of the letters: $\angle CBA$ and $\angle BDA$.

(*b*) $\angle DBC$ (*c*) $\angle ADC$ (*d*) $\angle 2$ or $\angle BDC$ (*e*) $\angle 3$ or $\angle ABD$

5.2 ADDING and SUBTRACTING ANGLES



Find (a) $\angle AOC$, (b) $\angle BOE$, (c) obtuse $\angle AOE$.

Solution:

(a) $\angle AOC = \angle a + \angle b = 90° + 35° = 125°$

(b) $\angle BOE = \angle b + \angle c + \angle d = 35° + 45° + 50° = 130°$

(c) $\angle AOE = 360° - (\angle a + \angle b + \angle c + \angle d) = 360° - 220° = 140°$

5.3 FINDING PARTS of ANGLES

Find (a) $\frac{2}{5}$ of a rt. $\angle$, (b) $\frac{2}{3}$ of a st. $\angle$, (c) $\frac{1}{2}$ of 31°, (d) $\frac{1}{10}$ of 70°20′.

Solution:

(a) $\frac{2}{5}(90°) = 36°$

(b) $\frac{2}{3}(180°) = 120°$

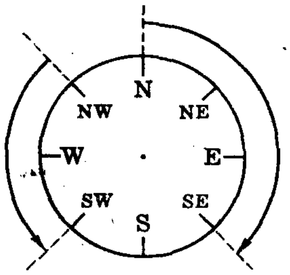
(c) $\frac{1}{2}(31°) = 15\frac{1}{2}° = 15°30'$

(d) $\frac{1}{10}(70°20') = \frac{1}{10}(70°) + \frac{1}{10}(20') = 7°2'$

5.4 FINDING ROTATIONS

In a half hour, what turn or rotation is made (a) by the minute hand, (b) by the hour hand?

What rotation is needed to turn: (c) from North to Southeast in a clockwise direction, (d) from Northwest to Southwest in a counterclockwise direction?



Solution:

(a) In one hour, a minute hand completes a full circle of 360°. Hence in a half hour it turns 180°.

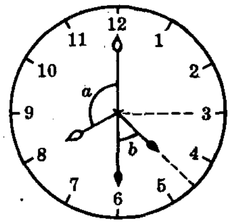
(b) In one hour, an hour hand turns $\frac{1}{12}$ of 360° or 30°. Hence in a half hour it turns 15°.

(c) Add turn of 90° from North to East and 45° from East to Southeast. $90° + 45° = 135°$.

(d) The turn from Northwest to Southwest is $\frac{1}{4}(360°) = 90°$.

5.5 FINDING ANGLES

Find the angle formed by the hands of the clock: (a) at 8 o'clock, (b) at 4:30 o'clock.



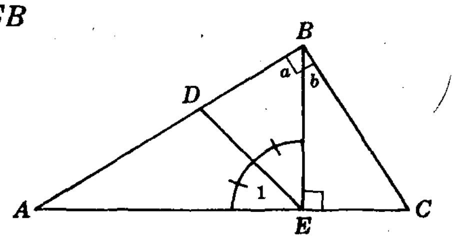
Solution:

(a) At 8 o'clock, $\angle a = \frac{1}{3}(360°) = 120°$.

(b) At 4:30 o'clock, $\angle b = \frac{1}{2}(90°) = 45°$.

5.6 APPLYING ANGLE FACTS

In the diagram shown: (a) Name two pairs of perpendicular lines, (b) find $\angle a$ if $\angle b = 42°$, (c) find $\angle AEB$ and $\angle CED$.



Solution:

(a) Since $\angle ABC$ is a right angle, $AB \perp BC$.
Since $\angle BEC$ is a right angle, $BE \perp AC$.

(b) $\angle a = 90° - \angle b = 90° - 42° = 48°$.

(c) $\angle AEB = 180° - \angle BEC = 180° - 90° = 90°$.
$\angle CED = 180° - \angle 1 = 180° - 45° = 135°$.

6. Triangles

A polygon is a closed plane figure bounded by straight line segments as sides. Thus Fig. (*a*) is a polygon. A polygon of 5 sides is a pentagon; it is named pentagon *ABCDE*, using its letters in order.

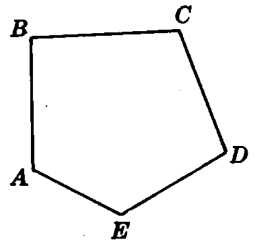


Fig.(*a*)

A triangle is a polygon having three sides. A vertex of a triangle is a point at which two of the sides meet. Vertices is the plural of vertex. The symbol for triangle is $\triangle$: for triangles, $\triangle\!\!\!\triangle$.

A triangle may be named using its three letters in any order or using a Roman numeral placed inside of it. Thus the triangle shown in Fig. (*b*) is named $\triangle ABC$ or $\triangle$I; its sides are *AB*, *AC* and *BC*; its vertices are *A*, *B* and *C*; its angles are $\angle A$, $\angle B$ and $\angle C$.

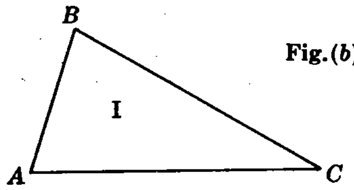


Fig.(*b*)

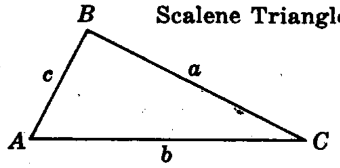
A. *Classifying Triangles:*

Triangles are classified according to the equality of their sides or according to the kind of angles they have.

Triangles According to the Equality of their Sides

1. **Scalene Triangle** – *A scalene triangle is a triangle having no equal sides.*

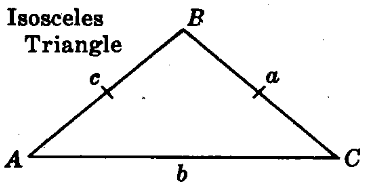
   Thus in scalene triangle *ABC*, $a \neq b \neq c$. The small letter used for each side agrees with the capital letter of the angle opposite it. Also, $\neq$ means "is not equal to".



Scalene Triangle

2. **Isosceles Triangle** – *An isosceles triangle is a triangle having at least two equal sides.*

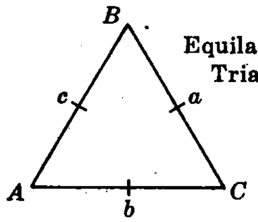
   Thus in isosceles triangle *ABC*, $a = c$. These equal sides are called the *legs or arms* of the isosceles triangle; the remaining side is the *base b*. The angles on either side of the base are the *base angles*; the angle opposite the base is the *vertex angle*.



Isosceles Triangle

3. **Equilateral Triangle** – *An equilateral triangle is a triangle having three equal sides.*

   Thus in equilateral triangle *ABC*, $a = b = c$. Note that an equilateral triangle is also an isosceles triangle.

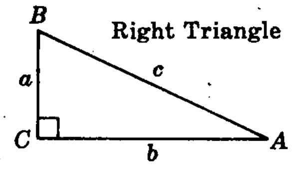


Equilateral Triangle

Triangles According to the Kind of Angles

1. **Right Triangle** – *A right triangle is a triangle having a right angle.*

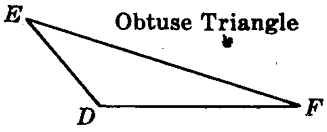
   Thus in right triangle *ABC*, $\angle C$ is the right angle. Side *c* opposite the right angle is the *hypotenuse*. The perpendicular sides, *a* and *b*, are the *legs or arms* of the right triangle.



Right Triangle

2. **Obtuse Triangle** – *An obtuse triangle is a triangle having an obtuse angle.*

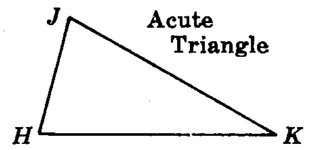
   Thus in obtuse triangle *DEF*, $\angle D$ is the obtuse angle.



Obtuse Triangle

3. **Acute Triangle** – *An acute triangle is a triangle having three acute angles.*

   Thus in acute triangle *HJK*, $\angle H$, $\angle J$ and $\angle K$ are acute angles.

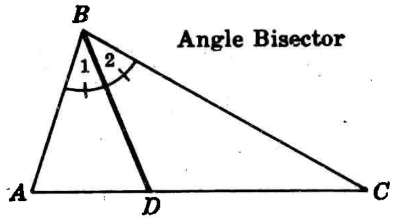


Acute Triangle

B. Special Lines in a Triangle

1. **Angle Bisector of a Triangle** — *An angle bisector of a triangle is a line that bisects an angle and extends to the opposite side.*

   Thus BD, the angle bisector of $\angle B$, bisects $\angle B$, making $\angle 1 = \angle 2$.

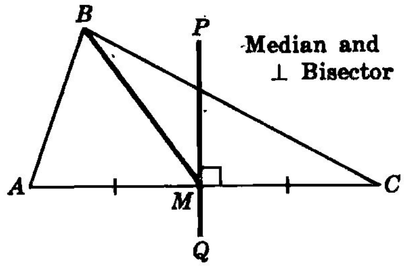


2. **Median of a Triangle. Perpendicular Bisector of a Side** — *A median of a triangle is a line from a vertex to the midpoint of the opposite side.*

   Thus BM, the median to AC, bisects AC, making $AM = MC$.

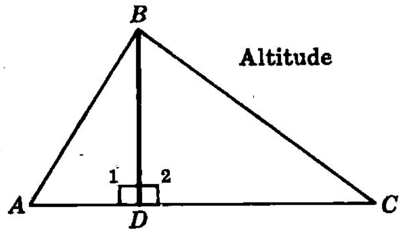
   A perpendicular bisector of a side of a triangle is a line that bisects and is perpendicular to a side.

   Thus PQ, the perpendicular bisector of AC, bisects AC and is perpendicular to it.



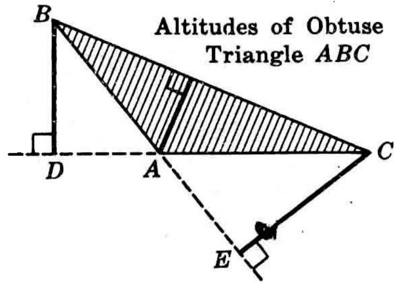
3. **Altitude to a Side of a Triangle** — *An altitude of a triangle is a line from a vertex perpendicular to the opposite side.*

   Thus BD, the altitude to AC, is perpendicular to AC and forms right angles 1 and 2. Each angle bisector, median and altitude of a triangle extends from a vertex to the opposite side.



4. **Altitudes of Obtuse Triangle** — *In an obtuse triangle, the altitude drawn to either side of the obtuse angle falls outside the triangle.*

   Thus in obtuse triangle ABC (shaded), altitudes BD and CE fall outside the triangle. In each case, a side of the obtuse angle must be extended.



6.1 NAMING a TRIANGLE and its PARTS

In Fig. 1, name (*a*) an obtuse triangle, (*b*) two right triangles and the hypotenuse and legs of each. (*c*) In Fig. 2, name two isosceles triangles; also name the legs, base and vertex angle of each.

Solution:

(*a*) Since $\angle ADB$ is an obtuse angle, $\triangle ADB$ or $\triangle$II is obtuse.

(*b*) Since $\angle C$ is a right angle, $\triangle$I and $\triangle ABC$ are right triangles. In $\triangle$I, AD is the hypotenuse and AC and CD are the legs. In $\triangle ABC$, AB is the hypotenuse and AC and BC are the legs.

(*c*) Since $AD = AE$, $\triangle ADE$ is an isosceles triangle. In $\triangle ADE$, AD and AE are the legs, DE is the base and $\angle A$ is the vertex angle.

Since $AB = AC$, $\triangle ABC$ is an isosceles triangle. In $\triangle ABC$, AB and AC are the legs, BC is the base and $\angle A$ is the vertex angle.

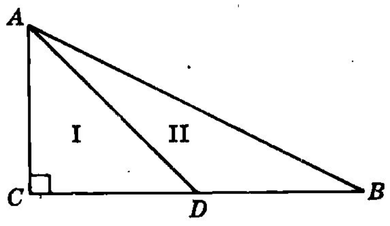


Fig. 1

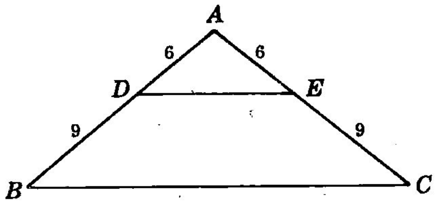


Fig. 2