Bubul Saikia

Fuzzy sets, Rough sets and Soft sets

Basic concepts and applications

VDM

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Chapter 1

Introduction

Fuzzy set theory provides a means for representing uncertainties and modeling the related concepts. Historically, probability theory has been the primary tool for representing uncertainty in mathematical models. But it deals with only random uncertainty. So, nonrandom uncertainties are not suited to treatment or modeling by probability theory. In fact, an overwhelming amount of uncertainty associated with complex systems and issues, which humans address on a daily basis, is nonrandom in nature. Fuzzy set theory is a marvelous tool for modeling nonrandom uncertainty, i.e., uncertainty associated with vagueness, with imprecision, with lack of detailed information regarding the problem at hand.

The notion of fuzzy sets provides an alternative approach to the traditional notions of set membership and logic whose roots lie in ancient Greek philosophy. Some philosophers of that time, including Aristotle, proposed 'laws of thought' in their effort to formulate a theory of logic (foundations of Mathematics).It contained 'the law of excluded middle' meaning that every proposition must either be 'true' or 'false'. But Plato claimed that there exists a third region between 'true' and 'false'. A central idea in his philosophy is that, in the real world, elements are very closely associated with imperfection and hence, there exists no element that is perfectly round. "Perfect notions" or "exact concepts" are the sort of things envisaged in pure mathematics while "inexact structure" prevail in real life. Fuzzy sets deal with situations using truth values which are true or false or ranging between 'true' and 'false'. The membership function of a fuzzy set maps each element of the universe to a value in the unit interval. The primary feature of fuzzy sets is that their boundaries are not precise. This facilitates the assignment of a subjective membership value to the elements of the universe of interest without totally rejecting or accepting them. This approach of subjective memberships taking all relevant aspect of the situation makes the fuzzy set quite user friendly. Fuzzy set theory does not handle the value of non membership value of its members explicitly. It is automatically determined by the difference between the membership value and unity. For example, we often here statements like "sixty percent of the voters voted in favour of a party". It does not mean the remaining forty percent of the voters have not voted for the particular party because there may be voters who have failed to vote due some reasons. In other words, a voter who has not voted in an election is not same as he has not voted for a particular party. Thus non membership value is not always automatically the difference between unity and the membership value. Considering this fact that the membership value always does not determine the non-membership value of an element, an extended definition of fuzzy sets was initiated by Prof. Atanassov **[5, 6]**. For a set, he defined a membership and a non membership function each of which maps each element of the set to a value in the unit interval with the constraint that sum of these two functional values must be less than or equal to unity. The remaining part (the difference of the sum of the membership and non membership from unity), if non zero, is the indeterminacy part of the evaluator's conception of the particular element. This set with a membership and non membership (each of which may be partial) is called an intuitionistic fuzzy set which is in fact a generalization of fuzzy set.

Rough set theory is a comparatively recent mathematical approach to formulate sets with imprecise boundaries. It represents a different mathematical approach to vagueness and uncertainty. Here we associate some information (data, knowledge) with every object in the universe of discourse such that objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated in this way is the mathematical basis of the rough set theory. In recent years, there has been a rapid growth of interest worldwide in rough set theory and its application. It has been seen that fuzzy sets and rough sets are two different topics [**30**] and none generalizes the other.

A fundamental axiom in the theory of sets is the action of extensionality [**50**].This action defines equality among sets. According to this axiom two sets A and B are equal, if for any $x \in A$ it follows that $x \in B$ and any $y \in B$ implies $y \in A$. Two important properties that follow from this axiom are that ordering of elements in a set does not matter or repeated elements in a set are redundant. A list provides a structure in which ordering counts. In some situations we may want a structure which is a collection of objects in the same sense as a set but in which redundancy counts. For example, a collection of objects corresponding to the ages of people in a company do have a redundancy which we may desire to make explicit in the underlying set. With this in mind, Yager [**93**] introduced the concept of a bag drawn from a set X and mentioned some possible applications of this bag structure in relational databases. He defined various operations on bags, introduced the concept of fuzzy bags and discussed some operations on them.

The theory of fuzzy sets developed by Zadeh [**100**] has established itself as quite an appropriate theory for dealing with uncertainties. But recently D. Molodtsov [**60**] showed in his paper 'Soft set theory – First Results' that fuzzy set theory has also some difficulties in handling uncertainties due to the inadequacy of the parametrization tool. Soft set theory is a new mathematical tool for dealing with uncertainties which is free from above difficulties. Also, Soft set theory has a rich potential for applications in several directions. Molodtsov has shown that fuzzy set might be considered as a special case of the soft set.

Before furnishing the summary of our results, we present a brief description of the theories of Fuzzy sets, Intuitionistic fuzzy sets, Rough sets, Bags, Fuzzy bags and Soft sets.

1.1 Fuzzy Sets

Fuzzy set theory proposed by Professor L.A.Zadeh at the University of California, Berkeley in 1965 is a generalization of classical or crisp sets. It makes possible to describe vague notions and deals with the concepts and techniques which lay in the form of mathematical precision to human thought processes that in many ways are imprecise and ambiguous within the ambit of classical mathematics. This theory reflects itself as a multi-dimensional field of inquiry, contributing to a wide spectrum of areas ranging from para-mathematical to human perception and judgment. It deals with situations using truth values ranging between the usual "true and false".

When vague notions arise, it is sometimes difficult to determine the exact boundaries of the class and hence the decision that whether an element belongs to it or not is replaced by a measure from scale. Each element of the class is evaluated by a measure which expresses its place and role in the class. This measure is called the grade of membership in the given class. A set in which each element of the universal set is characterized by its membership grade is called a fuzzy set.

Let X be a classical set of objects called the universe and x be any arbitrary element of X. Membership in a classical subset A of X is defined through a characteristic function from X to {0,1} such that

$$\mu_A(x)=\begin{cases}1 & \text{if } \quad x\in A\\ 0 & \text{if } \quad x\notin A\end{cases}$$

where {0, 1} is called a valuation set. If the valuation set is taken to be the real interval [0, 1] then A is called a fuzzy set. In the year 1967, Goguen [**33**] proposed a purely mathematical definition of fuzzy set by taking a more general poset as a valuation set instead of [0,1] as in Zadeh's definition and discussed in detail the case when this poset is a complete lattice ordered semigroup. In a fuzzy set A, $\mu_A(x)$ is called the grade of the membership of x in A. The closer is the membership value $\mu_A(x)$ to 1, the more x belongs to A. Thus, we can view A as a subset of X that has no sharp boundary and is completely characterized by the set of pairs $\{(x,\mu_A(x):x\in X\}$.The standard complement $\bar{A}$, of a fuzzy set A with respect to universal set X is defined by the characteristic function $\mu_{\bar{A}}(x)=1-\mu_A(x)$.

For fuzzy sets A, B, C and D on X, we have

A=B iff $\mu_A(x)=\mu_B(x), \quad \forall x \in X$;

$A \subseteq B$ iff $\mu_A(x) \le \mu_B(x), \quad \forall x \in X$;

$B \supseteq A$ iff $A \subseteq B$;

$C=A \cup B$ iff $\mu_C(x)=\max\{\mu_A(x),\mu_B(x)\}, \quad \forall x \in X$;

$D=A \cap B$ iff $\mu_D(x)=\min\{\mu_A(x),\mu_B(x)\}, \quad \forall x \in X$.

Note that the operations fuzzy union, fuzzy intersection and fuzzy complement contrary to their counterparts in case of crisp sets, are not unique, i.e., different functions may be used to represent these operations in different contexts. Therefore, like membership value of fuzzy sets, the operations of fuzzy sets are context dependent. The law of contradiction and the law of excluded middle are not satisfied by fuzzy set and fuzzy logic.

1.1.1 Fuzzy Relations

Fuzzy relations also relate elements of one universe, say X, with elements of another universe, say, Y, through the Cartesian product of two universes. But the strength of the relationship is not measured with the characteristic function, rather with a membership function expressing various "degrees" of the relation on the interval [0,1]. Thus the degree or grade of membership of a member in the relation R is $\mu_R(x,y) \in [0,1], \forall\ x \in X$ and $\forall\ y \in Y$. Here R can be considered as a fuzzy set in the universe X×Y.

Note that it is just a generalization of crisp relation. Another case of fuzzy relation is that maps Cartesian product of fuzzy sets A×B contained in the universal set X×Y into the unit interval [0,1].

A fuzzy relation R on a single universe X is also a relation from X to X. It is called a fuzzy equivalence relation if it satisfies the following three properties:

i. Reflexivity

ii. Symmetry

iii. Transitivity

A fuzzy tolerance relation is a fuzzy relation that satisfies only the reflexive and symmetric properties.

1.1.2 Similarity Measures

A similarity measure is a matching function to measure the degree of similarity between two fuzzy sets. This degree of similarity indicates the closeness of two sets and associates a numerical value with the idea that a higher value indicates a greater similarity. A number of similarity measures have been proposed in the literature for measuring the degree of similarity between fuzzy sets.

Let F(X) be the set of all fuzzy sets drawn from the set X and

S: $F(X)\times F(X)\rightarrow[0,1]$.Then S(A,B) is said to be the degree of similarity between the fuzzy sets A and $B\in F(X)$ if S(A,B) satisfies the following properties:

i. $0\leq S(A,B)\leq 1$;

ii. S(A,B)= 1 if A=B;

iii. S(A,B)= S(B,A).

1.1.3 Fuzzy Numbers

Fuzzy numbers model imprecise quantities (numbers) like 'about 10', 'below 100' etc. A fuzzy number is a quantity whose value is imprecise, rather than exact as in the case with 'ordinary' (single-valued) numbers. Any fuzzy number can be thought of as a function whose domain is a specified set (usually the set of real numbers, and whose range is the span of non-negative real numbers between 0 and 1(both inclusive).Each numerical value in the domain is assigned a specific 'grade of membership', where 0 represents the smallest possible grade, and 1 is the largest possible grade. In many respects, fuzzy numbers depict the physical world more realistically than single-valued numbers. Fuzzy numbers are used in statistics, computer programming, engineering (especially communications), and experimental science. The concept takes into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty.

1.1.4 Fuzzy Logic

Logic is the science of reasoning. Symbolic or mathematical logic has turned out to be a powerful computational paradigm. Not only does symbolic logic help in the description of events in the real world but also has turned out to be an effective tool for inferring or deducing information from a given set of facts. Just as mathematical sets have been classified into crisp sets and fuzzy sets, logic can also be broadly viewed as crisp logic and fuzzy logic. Also we have that crisp sets survive on a 2-state membership (0/1) and fuzzy sets on a multistate membership [0,1], similarly crisp logic is built on a 2-state truth value(true/false) and fuzzy logic on a multistate truth value(true/false/partly true/ partly false and so on). Fuzzy logic seems closer to the way our brains work. We aggregate data and form a number of partial truths which we aggregate further into higher truths. When certain thresholds are exceeded, these cause certain further results such as motor reaction. The ultimate goal of fuzzy logic is to form the theoretical foundation for reasoning about imprecise propositions and such reasoning has been referred to as approximate reasoning. Approximate reasoning is analogous to predicate logic for reasoning with precise propositions, and hence is an extension of classical predicate calculus for dealing with partial truths.

1.1.5 Fuzzification and Defuzzification

Fuzzification is the process of changing a crisp quantity into a fuzzy quantity. It can be done by simply recognizing that of the quantities which are considered to be crisp and deterministic are actually not deterministic at all. They carry considerable uncertainty. If the form of uncertainty happens to arise because of imprecision, ambiguity, or vagueness, then the variable is considered to be fuzzy and can be represented with the help of a membership function.

In many situations, where output is fuzzy, it is possible to take a crisp decision by converting the fuzzy output into a single scalar quantity. This conversion of a fuzzy set to a single crisp value is called defuzzification and involves the reverse process of fuzzification. There are several methods for defuzzification of which frequently need ones are centroid method, centre of sums and mean of maxima etc.

1.2 Intuitionistic Fuzzy Sets (IFS)

Out of several higher order fuzzy sets, intuitionistic fuzzy sets (IFS) introduced by Atanassov[**5**,**6**] have a lot of potential for applications. IFS are not fuzzy sets, although these are defined with the help of membership functions. But fuzzy sets are intuitionistic fuzzy sets. Atanassov [**5**] himself gave one example showing that fuzzy sets are intuitionistic fuzzy sets, but the converse is not necessarily true. There are many situations where intuitionistic fuzzy sets are more useful to deal with. Let E be the set of all states of India with elective governments. Assume that we know for every state $x \in E$ the percentage of the electorate who have voted for the corresponding government. Let it be denoted by M(x) and let $\mu(x)=\dfrac{M(x)}{100}$. Let $\nu(x)=1-\mu(x)$.This number corresponds to that part of electorate who have not voted for the government. By means of the fuzzy set theory we cannot consider this value in more detail. However, if we define $\nu(x)$ as the number of votes given to parties or persons outside the government, then the part of electorate who have not voted at all will have membership value $1-\mu(x)-\nu(x)$.Thus the resulting set, denoted by $\{< x,\mu(x),\nu(x)>|x\in E\}$ is called an intuitionistic fuzzy set [**5**]. Let a (non fuzzy) set E be fixed. An intuitionistic fuzzy set (IFS) A in E is defined as $A=\{< x,\mu(x),\nu(x)>\}|x\in E\}$, where the functions $\mu_A:E\rightarrow[0,1]$ and $\nu_A:E\rightarrow[0,1]$ define the degree of membership and degree of non membership respectively of the element x to the set A and for every $x\in E, 0\leq\mu_A(x)+\nu_A(x)\leq 1$.

1.3 Rough Sets

Rough sets have been introduced by Z.Pawlak in 1982 [**68**] to provide a systematic framework for studying imprecise and insufficient knowledge. It is a strong mathematical tool to deal with vagueness. For a very long time, philosophers and logicians have been attracted by the concept which is related to the so called boundary line view. These objects can be classified neither to the concept nor to its complement and thus there are boundary line cases. The underlying assumption behind the concept of rough sets is that knowledge has granular structure which is caused by the situation when some objects of interest cannot be distinguished as they may appear to be identical. The indiscernibility relation thus generated is the mathematical basis of Pawlak's rough sets.This can be used to describe dependencies between attributes, to evaluate significance of attributes and to deal with inconsistent data. The main advantage of rough set theory is that it does not need any preliminary or additional information about the data. According to the approach of rough set theory we first of all assume that any vague concept is characterized by a pair of precise concepts called the lower and the upper approximations of the concerned vague concept. The lower approximation is the set of objects surely belonging to the concept and the upper approximation is the set consisting of all objects possibly belonging to the concept. The boundary region of the vague concept is merely the difference between the upper and the lower approximations of it. Suppose R is an equivalence relation defined over the universe U which, in turn, partitions U into disjoint equivalent classes. Then for any subset X of U, the sets $\underline{A}(X)=\{x:[x]_R \subseteq X\}$ and $\overline{A}(X)=\{x:[x]_R \cap X \neq \phi\}$ are respectively called the lower and the upper approximation of X. The pair A=(U,R) is called the approximation space and $A(X)=(\underline{A}(X),\overline{A}(X))$ is called the rough set of X in U. In the above, $[x]_R$ denotes the equivalence class with respect to R containing x. Further, for a fixed non empty subset X of U, the rough set of X, i.e., A(X) is unique.

For any subset $X \subseteq U$ representing a concept of interest, the approximation space A=(U,R) can be characterized with three distinct regions of X : the so called positive region $\underline{A}$ (X), the boundary region $\overline{A}(X)-\underline{A}(X)$, and the negative region $U-\overline{A}(X)$ .The characterization of objects in X by the indiscernibility relation R is not precise enough if the boundary region $\overline{A}(X)-\underline{A}(X)$ is not empty. In such a case it may be impossible to say whether an object belongs to X or not, so that the set X is said to be non definable in A, and X is a rough set. For simplicity, we denote a rough set $A(X)=(\underline{A}(X),\overline{A}(X))$ of X by $A(X)=(\underline{X},\overline{X})$. Let $A(X)=(\underline{X},\overline{X})$ and $A(Y)=(\underline{Y},\overline{Y})$ be any two rough sets in the approximation space A=(U,R). Then (i) $A(X) \cup A(Y)=(\underline{X} \cup \underline{Y}, \overline{X} \cup \overline{Y})$

(ii) $A(X) \cap A(Y)=(\underline{X} \cap \underline{Y}, \overline{X} \cap \overline{Y})$

(iii) $A(X) \subset A(Y)$ iff $\underline{X} \subset \underline{Y}, \overline{X} \subset \overline{Y}$

(iv) The rough complement of A(X) in (U,R)

denoted by -A(X) and is defined by $-A(X)=(U-\underline{X}, U-\overline{X})$

(v) $A(X)-A(Y)=(\underline{X}-\overline{Y}, \overline{X}-\underline{Y})$.

1.4 Bags and Fuzzy Bags

Yager[**93**] introduced the bag structure as a set like object in which repeated elements are significant. A set generally implies a collection into a whole of definite well distinguished objects where redundant objects are not counted. In fact there are many collections like collection of books in a library, collection of medicine in a pharmacy, collection of zeros of an algebraic polynomial etc., which are not sets, but bags.The application and usefulness of bag in the real life situations is very important, especially in relational database[**93**], decision making[**28,73,74**]etc.

A bag (or crisp bag) B drawn from a set X is represented by a function $Count_B$ and is defined as C_B:X $\to$N, where N is the set of all non negative integers. The function C_B is called the count function of the bag and for each $x \in X$, the value $C_B(x)$ indicates the number of times (i.e., multiplicity) the element x appears in the bag B. The bag B is represented by $B=\{x/C_B(x) : x \in X\}$. For example, the bag B drawn from the set $X=\{x_1,x_2,.....,x_m\}$ is represented as $B=\{x_1/n_1, x_2/n_2,.....,x_m/n_m\}$, where n_i is the number of occurrences of the element x_i in the bag B, i.e., $n_i=C_B(x_i)$, i=1,2,...,n. In [**93**] Yager has proposed the operations of intersection, union, addition etc. on bags together with the operation of selection of elements from a bag and bag projection. In addition to these, he has also defined fuzzy bags (i.e., bags with fuzzy elements, in which an object(element) may appear with a number of different membership grades).Thus, a fuzzy bag F drawn from a set X is characterized by a function $CM_F : X \to B$, where B is the set of all bags drawn for the unit interval [0,1].Yager has also defined some operations on fuzzy bags such as the sum of fuzzy bags, removal of a fuzzy bag from another fuzzy bag, union and intersection of fuzzy bags etc.

1.5 Soft Sets

Molodtsov [**60**] pointed out that classical methods can not be successfully used to solve complicated problems in economics, engineering and environment because of various uncertainities typical of these problems. The important existing theories, viz., theory of probability, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics, theory of rough sets etc. can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties as pointed out in [**60**]. The reason for these difficulties is, possibly, the inadequacy of the parametrization tools of these theories. Molodtsov [**60**] introduced a new mathematical theory called 'Soft set' for dealing with uncertainties which is free from the above difficulties. Let U be an universe set and let E be a set of parameters. A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U, i.e., $F:E \rightarrow P(U)$, where P(U) is the power set of U. In other words, the soft set is a parametrized family of subsets of the set U. Every set F(e), for $e \in E$, from this family may be considered as the set of e–approximate elements of the soft set (F,E).For an illustration of soft set [**60**], suppose U = the set of houses available for purchase, E= the set of parameters whose each parameter is a word or a sentence, say expensive houses, beautiful houses, and so on. It is worth noting that the sets F(e) may be arbitrary, some of them may be empty, while some may have non empty intersection.

Zadeh's fuzzy set may be considered as a special case of the soft set. For this, let A be a fuzzy set, and μ_A be the membership function of the fuzzy set A, i.e., μ_A is a mapping of U into [0,1], where U is the universal set. Then $F(\alpha)=\{x \in U/\mu_A(x) \geq \alpha\}$, $\alpha \in [0,1]$ is a family of α-level sets of the function μ_A. If the family F is known, one can find the functions $\mu_A(x)$ by means of the following formulae:

$$\mu_A(x)=\sup_{\substack{\alpha\in[0,1] \\ x\in F(\alpha)}} \alpha$$

Thus, every Zadeh's fuzzy set A may be considered as the soft set (F, [0,1]). Again, let, (X, τ) be a topological space, that is, X is a set and τ is a topology (τ is a family of subsets of X, called the open sets of X). Then the family of open neighbourhoods T(x) of point x, where $T(x) = \{ V \in \tau : x \in V \}$, may be considered as the soft set $(T(x), \tau)$.

The way of setting (or describing) any object in the soft set theory principally differs from the way in which we use classical mathematics. In classical mathematics, we construct a mathematical model of an object and define the notion of the exact solution of the model. Usually the mathematical model is too complicated and we may not find the exact solution. So, in the second step we introduce the notion of the approximate solution and calculate that solution. In the soft set theory, we have the opposite