INTRODUCTION TO THE

Theory of Fuzzy Subsets

VOLUME I

A. Kaufmann

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Fundamental Theoretical Elements

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Foreword

The theory of fuzzy subsets is, in effect, a step toward a rapprochement between the precision of classical mathematics and the pervasive imprecision of the real world —a rapprochement born of the incessant human quest for a better understanding of mental processes and cognition.

At present, we are unable to design machines that can compete with humans in the performance of such tasks as recognition of speech, translation of languages, comprehension of meaning, abstraction and generalization, decisionmaking under uncertainty and, above all, summarization of information.

In large measure, our inability to design such machines stems from a fundamental difference between human intelligence, on the one hand, and machine intelligence, on the other. The difference in question lies in the ability of the human brain —an ability which present-day digital computers do not possess— to think and reason in imprecise, nonquantitative, fuzzy terms. It is this ability that makes it possible for humans to decipher sloppy handwriting, understand distorted speech, and focus on that information which is relevant to a decision. And it is the lack of this ability that makes even the most sophisticated large scale computers incapable of communicating with humans in natural —rather than artificially constructed— languages.

The fundamental concept in mathematics is that of a set — a collection of objects. We have been slow in coming to the realization that much perhaps most, of human cognition and interaction with the outside world involves constructs which are not sets in the classical sense, but rather "fuzzy sets" (or subsets), that is, classes with unsharp boundaries in which the transition from membership to nonmembership is gradual rather than abrupt. Indeed, it may be argued that much of the logic of human reasoning is not the classical two-valued or even multivalued logic but a logic with fuzzy truths, fuzzy connectives, and fuzzy rules of inference.

In our quest for precision, we have attempted to fit the real world to mathematical models that make no provision for fuzziness. We have tried to describe the laws governing the behavior of humans, both singly and in groups, in mathematical terms similar to those employed in the analysis of inanimate systems. This, in my view, has been and will continue to be a misdirected effort, comparable to our long-forgotten searches for the perpetuum mobile and the philosopher's stone.

What we need is a new point of view, a new body of concepts and techniques in which fuzziness is accepted as an all pervasive reality of human existence. Clearly, we need an understanding of how to deal with fuzzy sets within the framework of classical mathematics. More important, we have to develop novel methods of treating fuzziness in a systematic—but not necessarily quantitative— manner. Such methods could open many new frontiers in psychology, sociology, political science, philosophy, physiology, economics, linguistics, operations research, management science, and other fields, and provide a basis for the design of systems far superior in artificial intelligence to those we can conceive today.

Professor Kaufmann's work is a highly important contribution to the attainment of these objectives. With his characteristic thoroughness and lucidity, he has given us a first systematic exposition of a subject area that, in the years to come is likely to have a significant impact on the orientation of science and engineering. Reflecting his exceptionally broad expertise in a wide variety of areas in applied mathematics, system theory, and engineering, Professor Kaufmann's treatment of fuzzy sets casts much light on their theory and enables him to extend it in many new and important directions.

The present volume is concerned, in the main, with the mathematical aspects of the theory of fuzzy sets. The volume to follow will cover the applications to problem areas centering on information and decision processes in humans and machines. In such applications, the concept of a fuzzy algorithm plays a central role and the existence of fuzzy feedback makes it possible to deal with fuzzy sets in a qualitative manner closely approximating the nonquantitative thought processes of the human brain.

Professor Kaufmann's treatise is clearly a very important accomplishment. It may well exert a significant influence on scientific thinking in the years ahead and stimulate much further research on the theory of fuzzy sets and their applications in various fields of science and engineering.

L.A. ZADEH
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Preface

In an epoch in which the elite make science their guide while the masses return to magic, may one say that science is objective, without weaknesses for inexactitude? One is, in many scientific disciplines, more careful to make distinctions. Certainly, objective knowledge must be the purpose of human evolution, but a permanent modesty must be present in the researcher and in the engineer; we know nothing of reality other than through our models, our representations, our more or less true laws, our acceptable approximations in the state of our knowledge. And the model of something for one is not exactly the same model of this thing for another; the formula may remain the same, but the interpretation may be different. The universe is perceived with the aid of models that are indeed perfecting themselves through embodying one in another, at least until some revolution in ideas appears, no longer permitting a correct embodyment.

Our models are fuzzy; our thoughts formed from more or less independent models are fuzzy; in such a manner are we different from a computer! A computer is a logical sequential machine that may not have, by the nature of its definition, any theoretical error; it is a nonfuzzy machine by definition. But man possesses, in addition to the faculty of reckoning and thinking logically, that of taking things into account globally or in parallel, as all living beings. This global or parallel reasoning, as opposed to logical reckoning, is fuzzy and must be fuzzy. A living being, having the possibility of initiative, perceives and treats a piece of more or less fuzzy information and adapts itself. When the living being has almost no more initiative, when its entropy function is almost zero, the fuzziness may then disappear; the being is programmed. A cell, in biology, functions as a small computer commanding a small factory (the word *small* applies here to size but not complexity); there is almost no entropy in this system. A man, on the contrary, has an immense entropy function; he may choose, decide, evolve, err, set right, begin again, understand a little, and build his knowledge in the adventure of science, with a formal program.

How does one join conceptually global reasoning and logical reasoning; how does one associate that which is physically true and that which is an interpretation of human thought in order to be close to both at the same time? How does one introduce fuzziness into mathematics, since it is finally in this most clear form that it will be necessary to express this at first strange association.

For a mathematician, what does the word *fuzzy* signify (or synonymous words)? This will mean that an element is a member of a subset only in an uncertain fashion; while, on the other hand, in mathematics we understand that there are only two acceptable situations for an element: being a member of or not being a member of a subset. Any formal logic, boolean logic, rests on this base: membership or nonmembership in a subset of a reference set.

The merit of L. A. Zadeh has been to attempt to leave this impasse by introducing the notion of weighted membership. An element may then belong more or less to a subset, and, from there, introducing a fundamental concept, that of a fuzzy subset.

In an obviously different presentation, constructed on n-ary logic, Post (1921), Lukasiewicz (1937), and Moisil (1940) have given general theories in which the theory of fuzzy subsets may be placed in a number of its aspects. It may be considered that the two schools have converged as expressed by the American author Zadeh and the Romanian author Moisil.

Speaking at various conferences on the theory of fuzzy subsets, I have frequently met the same proposition: what may be done with this theory may just as well be done without it. But this is true for any theory. I recall meeting such a proposition twenty years ago when I wrote with one of my friends one of the first works on matrix calculus; it was criticized as another book on tensor calculus. This criticism has been offered for graph theory, for applied modern mathematics, and for many other things. Perhaps those who say this have not understood to what uses mathematics is put, more or less independently of the fact that it constitutes a science in a pure state, that is, the science of sciences. Mathematics is our means of access to knowledge through logical models; their providing practical application is the explanation; they give us structure, and numbers are only totally ordered structures, a small particular, very convenient case of an infinite set of structures.

The theory of fuzzy subsets at least allows the structuring of all that which is separated by frontiers only a little precise, as thought, language, and perception among men. The humanistic sciences are replete with all sorts of abstract or concrete forms; and the sciences said to be exact also may be concerned with situations where uncertainty is in the nature of things. This relatively new theory is useful and important; scientists of all disciplines ought to be interested, but also the literati and artists, those who construct truth and beauty with fuzziness, with an indispensable fuzziness, which allows all nuances and which is the stimulant of the imagination, with this fuzziness that one may call mental entropy.

I have done my best to make this work accessible to the usual readers of my books: engineers, designers, professors, students, executives, decision makers. This theory is not always easy, it falls far short of it. The pedagogical effort will have its reward if this work compiling various personal works excites the reader to publish on the subject of this work applications and new ideas. I have asked Professor Zadeh to offer a foreword for this book; his support and his encouragement have been extremely important for the mathematician-engineer and author who has carried out this modest work.

As all my books, this one is didactic; and I have introduced examples throughout; this has lengthened the text, the number of pages, and as a consequence, increased the price of the work. This didactic aspect seems to me to be indispensable for the readership imagined for this book: the tens of thousands of engineers in the world. These people have a good background and good training for understanding and using mathematics; they appreciate that it facilitates their tasks. Of course, this makes the text, in its setting, somewhat heavy; and any professional mathematician who reads my book will often find it drawn and long. The mathematician likes explicit aesthetics; but the engineer, if he is sensitive to elegance, is not concerned only with the mathematics, he has other fish to fry. My experience as a professor, in more than forty countries where international organizations and the university have allowed me to go, and my experience as an author encourage me to continue on this didactic path where the objective is to attain in others knowledge of his own research efforts. Thus various flaws that are often consequences of these intentions ought to be pardonned.

Of course, the notion of fuzziness may be considered from various points of view, in the specification of variables and of their values, in configurations, and more generally at the conceptual level. The present theory is limited to variables and configurations, but one may already envisage how to attack the conceptual aspect, at the cost of some difficulties as one may guess. The humane sciences, and biology in particular, essentially pose fundamental questions concerning conceptual fuzziness. How does one describe mechanisms where almost everything depends on almost everything else, where the smallest element has its role, where all is based on information, messages, and their treatment? The theory of fuzzy subsets is a limited method for seeing these things; the future will reveal whether it is a rather large and interesting base or whether it is only a provisionary way of treating uncertainty or an aspect of uncertainty. This theory is surely only a subtheory of something much larger, as we shall see. Perhaps it would be better to attend to and propose more ambititious theories. We begin by attacking an elementary aspect of fuzziness, that concerning variables and configurations; this is a step forward. So many disciplines, economists, linguists, information theorists, biologists, psychologists, sociologists, etc., are involved with conceptual fuzziness that we will advance boldly. A multitude of work is to be found here and in the worlds between these disciplines. I am optimistic on this subject, but as all those who take a deep interest in these researches, this optimism is tempered with an attitude of prudence and patience.

An important point deserves to be discussed with emphasis in this preface. Will it be possible to treat these fuzzy problems with computers that are sequential machines using binary logic? The answer, as we shall see, is affirmative given a new hardware, a new software, or what is the same a new mixture of kinds of firmware. Thus this will change several habits among analysts and programmers, while very probably and soon the designers of technologies will include fuzzy logic circuits (how poorly the two words go together—we shall explain later), giving binary logical elements, semi n-ary logics or degree logics in new classes of computers. It is more and more evident and necessary that the man–machine dialogue must be put within reach of all, and no longer only by passing through formal languages and very simplified programming. And the language of man as his thoughts is fuzzy or/and logical; between him and the computer, intermediate stages, finite automata of all kinds, must permit better communication. Thus, we may predict without great risk of error that technologies will evolve toward an incorporation of these concepts closer to the usual thought of man, this while waiting for the day when machines other than computers will treat global information, treating it in parallel intrinsically without necessarily passing through a sequential treatment. Then these machines, which may be called combiners or parallel processors, will allow treating fuzziness through fuzziness and no longer treating fuzziness with the same microminature binary logic. Utopia today, reality in the next ten years. Artificial intelligence such as one has been able to conceive of until now with the aid of autoadaptive programs on computers will more closely approximate the intelligence of men (but is this then human intelligence?).

To comment on another point of detail, why do we use the term fuzzy subset instead of fuzzy set? This is because a fuzzy set will never be a concept proper to the present theory; the reference set will always be an ordinary set, that is, such as one defined intuitively in modern mathematics, that is again, a collection of well-specified and distinct objects. It is the subsets that will be fuzzy, as we shall see. I am not a stubborn and rigorous bourbakiste, those who have read one of my books will know better, but there are some definitions that must not be fuzzy. There are times when it is convenient to have a touch of the Bourbaki!

Volume I constitutes the first part of this work and contains the theoretical bases. Volume II contains applications with principal titles: fuzzy languages, fuzzy systems, fuzzy automata, fuzzy algorithms, machines and control, decision problems in a fuzzy universe, recognition of forms, problems of classification and selection, documentary research, etc. Volume II will be presented in the same didactic spirit with very numerous examples and in addition a detailed review of basic knowledge on each subject treated. While waiting for the second volume, the reader is referred to the bibliography where the cited articles specify well the first applications under consideration.

Several persons have given me their cooperation in reading the manuscript and offering constructive criticisms: Mme. Monique Peteau, Docteur en Sciences and MM. Michel Cools and Thierry Dubois, Ingenieurs de Recherche au Centre I.M.A.G.O. de l'Universite de Louvain; M. Etienne Pichat, Professeur a l'Institut d'Informatique d'Entreprise, Conservatoire National des Arts et Metiers; M. Jules Kun, Conseiller Scientifique a la Cie. Honeywell Bull; M. Combe, Ingenieur a la Cie. Honeywell Bull; M. Arnaud Henri-Labordere, Ingenieur a la Societe d'Economie et de Mathematiques Appliquees.

My son Alain, as usual, has borne the burden of corrections. He is a student of medicine who thinks, as do others, that medicine is at the same time an art and a science, and that mathematics and thus information are precise instruments that it will also be necessary for him to be able to use tomorrow.

It is intuitive but pretentious to affirm that the human mind is not a simple mechanism, reducible to more or less complicated programs; Kurt Gödel has demonstrated this formally; then we reconcile our desires for rigor and for imagination. Since it is impossible to construct a program of all programs, today or tomorrow, we remain fuzzy and creative.

For this second edition, I have rectified several errors and omissions that have been indicated to me by several readers. In several places I have modified the terminology.

A little before publication of the second edition of Volume I, Volumes II and III concerning the applications have appeared, several months in advance of the predicted dates. This very great effort of reading, compilation, of imagination, and of publication has already found its reward in the reception given to Volume I by my readers and friends. In a soil if fertile, ideas germinate, and indeed all the humanistic sciences will be better modeled with this mathematics of nuances and of subjectivity.

A. Kaufmann

List of Principal Symbols

A	ordinary set or subset
$\underset{\sim}{A}$	fuzzy subset
$\in$	symbol of membership
$\notin$	symbol of nonmembership
$\lvert A \rvert$, card A	number of elements or cardinality of A
$\subset$	is a subset of (inclusion)
$\subset\subset$	is a strict subset of (one also says, a true subset of)
$\not\subset$	noninclusion
$\cup$	union
$\cap$	intersection
$\overline{A}$	complement of A
$\overline{\underset{\sim}{A}}$	pseudo-complement of $\underset{\sim}{A}$
$\mathcal{P}(E)$	set of ordinary subsets of E, the power set of E
ϕ	empty subset
L^E	set of fuzzy subsets of E when the membership function takes its values in L. In certain cases this set is also denoted $\underset{\sim}{\mathcal{P}}(E)$
$E_1 \times E_2$	cartesian product or product of E_1 and E_2
$\Rightarrow$	metaimplication (one also says, usually but improperly, implication)
$\Leftrightarrow$	logical equivalence
$\exists x$	existential quantifier (there exists an x)
$\exists! x$	unique existential quantifier (there exists one and only one x)
$\forall x$	universal quantifier (for all x)
$E_1 \overset{\Gamma}{\rightsquigarrow} E_2$	a mapping Γ of E_1 into E_2
$E_2 \overset{\Gamma^{-1}}{\rightsquigarrow} E_1$	the mapping Γ^{-1} of E_2 into E_1 (inverse mapping of Γ)
$\Gamma_2 \circ \Gamma_1$	composition of the two mappings Γ_1 and Γ_2
iff	if and only if
$\oplus$	disjunctive sum
$X \preccurlyeq Y$	order relation
$X \prec Y$	strict order relation
N	set of natural numbers, N = {0, 1, 2, 3, . . .}
N_0	the set N but excluding 0
Z	set of integers, Z = {0, +1, −1, +2, −2, +3, −3, . . .}
Z_0	the set Z excluding 0
R	set of real numbers
R_0	the set R excluding 0
R^+	set of nonnegative real numbers

$\mathbf{R}_0^+$	set of positive real numbers
$\mathbf{R}^n$	the set product $\mathbf{R} \times \mathbf{R} \times \cdots \times \mathbf{R}$ (n factors) or the real space with dimension n
$]a, b[$	interval of **R** "open on the left and on the right," thus, $\{x \mid a < x < b\}$
$]a, b]$	interval of **R** "open on the left and closed on the right," thus, $\{x \mid a < x \leqslant b\}$
$[a, b[$	interval of **R** "closed on the left and open on the right," thus, $\{x \mid a \leqslant x < b\}$
$[a, b]$	interval of **R** "closed on the left and on the right," thus, $\{x \mid a \leqslant x \leqslant b\}$; one also says *segment*
$\mu_{\underset{\sim}{A}}(x)$	membership function for the element x with respect to the fuzzy subset $\underset{\sim}{A}$
$d(\underset{\sim}{A}, \underset{\sim}{B})$	generalized Hamming distance between two fuzzy subsets $\underset{\sim}{A}$ and $\underset{\sim}{B}$
$\delta(\underset{\sim}{A}, \underset{\sim}{B})$	generalized relative Hamming distance between two fuzzy subsets $\underset{\sim}{A}$ and $\underset{\sim}{B}$
$\underset{\approx}{A}$	the ordinary subset nearest to the fuzzy subset $\underset{\sim}{A}$
$\nu(\underset{\sim}{A})$	index of fuzziness of the fuzzy subset $\underset{\sim}{A}$
A_α	ordinary subset of level α of a fuzzy subset $\underset{\sim}{A}$
max(X, Y) or $X \vee Y$	maximum of X and Y
min(X, Y) or $X \wedge Y$	minimum of X and Y
sup(X, Y) or $X \triangledown Y$	limit superior of X and Y
inf(X, Y) or $X \vartriangle Y$	limit inferior of X and Y
$\hat{+}$	symbol for an algebraic sum, $a \hat{+} b = a + b - a \, . \, b$
$\underset{\sim}{G} \subset E_1 \times E_2$	fuzzy graph
$\underset{\sim}{\mathcal{R}}_2 \circ \underset{\sim}{\mathcal{R}}_1$	composition of two fuzzy relations
$\underset{\approx}{\mathcal{R}}$	ordinary relation nearest to a fuzzy relation $\underset{\sim}{\mathcal{R}}$
$\mu_{\underset{\sim}{B}}(y \parallel x)$	membership function of a conditioned fuzzy subset
$\mu_{\underset{\sim}{\mathcal{R}}}(x, y)$	membership function of the ordered pair (x, y) for the fuzzy relation $x \, \underset{\sim}{\mathcal{R}} \, y$
$\mathcal{R}^n$	represents $\mathcal{R} \circ \mathcal{R} \circ \cdots \circ \mathcal{R}$ (n times)
$\overline{\underset{\sim}{\mathcal{R}}}$	complementary relation of $\underset{\sim}{\mathcal{R}}$ such that $\mu_{\overline{\underset{\sim}{\mathcal{R}}}}(x, y) = 1 - \mu_{\underset{\sim}{\mathcal{R}}}(x, y)$
$\hat{\underset{\sim}{\mathcal{R}}}$	transitive closure of $\underset{\sim}{\mathcal{R}}$
$l(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$	value of a path from x_{i_1} to x_{i_r}
$l^*(x_i, x_j)$	strongest path from x_i to x_j
$\underset{\sim}{a}, \underset{\sim}{b}, \ldots$	fuzzy variables
$\underset{\sim}{f}(\underset{\sim}{a}, \underset{\sim}{b}, \ldots)$	function of fuzzy variables
$(E, *)$	groupoid
$\mathcal{D}(X_i, X_j)$	distance between two elements X_i and X_j
MOR(X, Y)	set of morphisms of a category
$\underset{\sim}{\Gamma}$	fuzzy mapping

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CHAPTER I

FUNDAMENTAL NOTIONS

1. INTRODUCTION

In this first chapter we review the principal definitions and concepts of the theory of ordinary sets, that is, those that are at the foundation of present-day mathematics; but these definitions and concepts will be reexamined and extended to notions that pertain to fuzzy subsets.

We shall progress rather slowly so that the reader who is not a mathematician but rather a user of mathematics will be able to follow without difficulty.

The examples will allow the reader to verify, step by step, whether the new notions have been well understood. But all that is presented in this first chapter is very simple; the difficulties will appear later.

The theory of ordinary sets is a particular case of the theory of fuzzy subsets (we shall see presently why it is necessary to say *fuzzy subset* and not *fuzzy set*—the reference set will not be fuzzy). We have here a new and very useful extension; but, as we shall note several times, what may be described or explained with the theory of fuzzy subsets may also be considered without this theory, using other concepts. One may always replace one mathematical concept with another. But will it be so clear or generative of properties that are easier to discover and prove, or to use?

2. REVIEW OF THE NOTION OF MEMBERSHIP

Let **E** be a set and **A** a subset of **E**:

(2.1) $$\mathbf{A} \subset \mathbf{E}.$$

One usually indicates that an element x of **E** is a member of **A** using the symbol $\in$:

(2.2) $$x \in \mathbf{A}.$$

In order to indicate this membership one may also use another concept, a characteristic function $\mu_{\mathbf{A}}(x)$, whose value indicates (yes or no) whether x is a member of **A**:

(2.3) $$\begin{aligned} \mu_{\mathbf{A}}(x) &= 1 \quad \text{if} \quad x \in \mathbf{A} \\ &= 0 \quad \text{if} \quad x \notin \mathbf{A}. \end{aligned}$$

Example. Consider a finite set with five elements:

(2.4) $$\mathbf{E} = \{x_1, x_2, x_3, x_4, x_5\}$$

and let

(2.5) $$\mathbf{A} = \{x_2, x_3, x_5\}.$$

And one writes

(2.6) $$\mu_{\mathbf{A}}(x_1) = 0, \mu_{\mathbf{A}}(x_2) = 1, \mu_{\mathbf{A}}(x_3) = 1, \mu_{\mathbf{A}}(x_4) = 0, \mu_{\mathbf{A}}(x_5) = 1.$$

This allows us to represent **A** by accompanying the elements of **E** with their characteristic-function values:

(2.7) $$\mathbf{A} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1)\}.$$

Recall the well-known properties of a boolean binary algebra:

Let $\bar{\mathbf{A}}$ be the complement of **A** with respect to **E**:

(2.8) $$\mathbf{A} \cap \bar{\mathbf{A}} = \phi,$$

(2.9) $$\mathbf{A} \cup \bar{\mathbf{A}} = \mathbf{E}.$$

(2.10) If $x \in \mathbf{A}$, $x \notin \bar{\mathbf{A}}$, and one writes

(2.11) $$\mu_{\mathbf{A}}(x) = 1 \text{ and } \mu_{\bar{\mathbf{A}}}(x) = 0.$$

Considering the example in (2.6) and (2.7), one sees:

(2.12) $$\mu_{\bar{\mathbf{A}}}(x_1) = 1, \mu_{\bar{\mathbf{A}}}(x_2) = 0, \mu_{\bar{\mathbf{A}}}(x_3) = 0, \mu_{\bar{\mathbf{A}}}(x_4) = 1, \mu_{\bar{\mathbf{A}}}(x_5) = 0,$$

and one writes

(2.13) $$\bar{\mathbf{A}} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 1), (x_5, 0)\}.$$

Given now two subsets **A** and **B**, one may consider the intersection

(2.14) $$\mathbf{A} \cap \mathbf{B}.$$

One has

(2.15) $$\begin{aligned} \mu_{\mathbf{A}}(x) &= 1 \quad \text{if} \quad x \in \mathbf{A} \\ &= 0 \quad \text{if} \quad x \notin \mathbf{A}, \end{aligned}$$

(2.16) $$\begin{aligned} \mu_{\mathbf{B}}(x) &= 1 \quad \text{if} \quad x \in \mathbf{B} \\ &= 0 \quad \text{if} \quad x \notin \mathbf{B}, \end{aligned}$$

(2.17) $$\begin{aligned} \mu_{\mathbf{A}\cap\mathbf{B}}(x) &= 1 \quad \text{if} \quad x \in \mathbf{A} \cap \mathbf{B} \\ &= 0 \quad \text{if} \quad x \notin \mathbf{A} \cap \mathbf{B}. \end{aligned}$$

This allows us to write

(2.18) $$\mu_{\mathbf{A}\cap\mathbf{B}}(x) = \mu_{\mathbf{A}}(x) \cdot \mu_{\mathbf{B}}(x),$$

where the operation . corresponds to the table in Figure 2.1 and is called the boolean product.

(.)	0	1
0	0	0
1	0	1

FIG. 2.1

In the same fashion for the two subsets A and B, one defines the union or join:

(2.19)
$$\mu_{A \cup B}(x) = 1 \quad \text{if} \quad x \in A \cup B$$
$$= 0 \quad \text{if} \quad x \notin A \cup B\,;$$

with the property

(2.20)
$$\mu_{A \cup B}(x) = \mu_A(x) \dotplus \mu_B(x),$$

where the operation $\dotplus$, the boolean sum, is defined by the table in Figure 2.2.

$(\dotplus)$	0	1
0	0	1
1	1	1

FIG. 2.2

Example. Consider the reference set (2.4) and the two subsets

(2.21)
$$A = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1)\},$$

(2.22)
$$B = \{(x_1, 1), (x_2, 0), (x_3, 1), (x_4, 0), (x_5, 1)\}.$$

One sees

(2.23)
$$A \cap B = \{(x_1, 0.1), (x_2, 1.0), (x_3, 1.1), (x_4, 0.0), (x_5, 1.1)\}$$
$$= \{(x_1, 0), (x_2, 0), (x_3, 1), (x_4, 0), (x_5, 1)\}.$$

(2.24)
$$A \cup B = \{(x_1, 0 \dotplus 1), (x_2, 1 \dotplus 0), (x_3, 1 \dotplus 1), (x_4, 0 \dotplus 0), (x_5, 1 \dotplus 1)\}$$
$$= \{(x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1)\}.$$

To continue, emanating from these two subsets one has

(2.25)
$$\overline{A \cap B} = \{(x_1, 1), (x_2, 1), (x_3, 0), (x_4, 1), (x_5, 0)\},$$

(2.26)
$$\overline{A \cup B} = \{(x_1, 0), (x_2, 0), (x_3, 0), (x_4, 1), (x_5, 0)\}.$$

These few exercises constitute only a didactic preamble to an understanding of fuzzy subsets.