MLM 4

Morningside Lectures in Mathematics

Lectures on the Analysis of Nonlinear Partial Differential Equations Vol. 4

非线性偏微分方程分析讲义 第四卷

○ Editors Jean-Yves Chemin
Fanghua Lin
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Preface to the First Volume

In summer of 2001, we initiated a summer school program on the harmonic analysis and its applications in nonlinear partial differential equations, with special emphases on nonlinear Schrödinger equations, kinetic equations of Bolzmann type and classical fluid equations. Over the years, there have been many distinguished mathematicians working in these fields who have come to help our program and to give series of special lectures. The program has been shown to be particularly helpful to young researchers and students. The lectures involved have gradually turned into more formal and regular seminars on *Analysis in Partial Differential Equations* at the Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences.

From June 2007 to January 2008, we held a special semester PDE–program. We invited many mathematicians and experts in mathematical theory of fluid mechanics and quantum mechanics. The visitors during that period include: Bresch Didier, Carles Rémi, Jean-Yves Chemin, Desvillettes Laurent, Lopes Filho Milton C, Nussenzveig Lopes Helena J., Novotny Antonin, Chao-Jiang Xu, Chongchun Zeng, Ping Zhang and Yuxi Zheng, who gave a series of lectures and provided excellent lecture notes. It is no doubt that these lecture notes would be very useful for many researchers and students. In this volume, we have collected lecture notes by M. C. Lopes concerning the boundary layers of incompressible fluid flow; by C. J. Xu on the micro-local analysis and its applications to the regularities of kinetic equations; by Y. X. Zheng on the weak solutions of variational wave equation from liquid crystals; and by P. Zhang and Z. F. Zhang on the free boundary problem of Euler equations. In addition, we also included the notes by F. Nier on the hypoellipticity of Fokker-Planck operator and Witten-Laplace operator that were given earlier in the summer of 2006.

We have planned to publish in the forthcoming volumes the other lecture notes. Some are from past lectures at our program and some will be collected from the newly scheduled seminars. We hope that the publication of these lecture notes may provide valuable references and up-to-date descriptions of current developments of various and related research topics, that will benefit many young

researchers or graduate students. We wish to take this opportunity to thank the Morningside Center of Mathematics, the Institute of Mathematics of AMSS that provides all necessary supports. We are particularly grateful to professor Lo Yang for his constant help, supports and encouragements to our program. We also would like to thank Guilong Gui for his careful preparations of the Latex file of the entire book. We finally appreciate for the financial support from the Chinese Academy of Sciences.

Fanghua Lin in New York
Xueping Wang in Nantes
Ping Zhang in Beijing
November 3, 2008

Preface to the Fourth Volume

This book is a sequel to the previous volumes *Lectures on the Analysis of Nonlinear Partial Differential Equations*, Vol. I, Vol. II, and Vol. III. In this fourth volume we have collected those lecture notes by Jean-Yves Chemin, Hongjie Dong, Xiaochun Li, Fanghua Lin, Alexis F. Vasseur, Jiahong Wu, and Xiaoyi Zhang. We appreciate very much their time and efforts to provide us with these excellent notes. We believe these lecture notes will serve as valuable references on current developments in various research topics in nonlinear partial differential equations (PDE).

The notes collected here came initially from the seminars on Analysis in Partial Differential Equations. It was a part of the PDE program (from April through October 2012) at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences (CAS). We would like to take this opportunity to thank the National Center for Mathematics and Interdisciplinary Sciences, Academy of Mathematics and Systems Science as well as the Morningside Center of Mathematics of CAS for the financial support. We thank also several other distinguished visitors who had delivered excellent lectures at the program: Hammadi Abidi, Albert Cohen, Marius Paicu, Benoit Perthame and Jean-Claude Saut. We wish their notes would be included in our future volumes.

Finally, we wish to thank Prof. Lo Yang, Prof. Nanhua Xi, and Prof. Yuefei Wang, for their continuing support over all these years which was essential to the success of this PDE program. We also thank Dr. Guilong Gui for the careful preparation of the Latex files that make up this entire book.

Jean-Yves Chemin in Paris
Fanghua Lin in New York
Ping Zhang in Beijing
June, 2015

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Profile Decomposition and Its Applications to Navier-Stokes System

Jean-Yves Chemin*

Abstract

The purpose of theses lectures is the description, up to an extraction, of lack of compactness of the Sobolev embeddings as achieved in the work [18] of P. Gérard. We apply this result to the description of the possible blow up in the three dimensional incompressible Navier-Stokes.

Introduction

This text is notes of a series of lectures given first in the Morningside Center of Mathematics in February and March 2014 in Beijing and then at Jacques-Louis Lions laboratory in May and June 2014. I thank very warmly the audience of this two series of lectures for their interest remarks. I want to thank especially Professor Ping Zhang for his rereading of the notes and a lot of suggestions of improvements.

The purpose of theses lectures was the description, up to an extraction, of lack of compactness of the Sobolev embeddings as achieved in the work [18] of P. Gérard. We apply this result to the description of the possible blow up in the three dimensional incompressible Navier-Stokes.

More precisely, the first section will be devoted to the proof of a refined Sobolev inequality which is in particular invariant under translation in Fourier space (i.e. multiplication by an oscillating function). This involves some particular classes of Besov spaces we shall define and study.

The second section is devoted to the statement and the detailed proof of the P. Gérard's result which provides a precise description, up an extraction, of a sequence of function which is bounded in the homogeneous Sobolev $\dot{H}^s(\mathbb{R}^d)$. It claims that in some sense, it is the sum of dilation and translation of some

*Laboratoire J.-L. Lions, Case 187, Université Pierre et Marie Curie, 75 230 Paris Cedex 05, France. E-mail: chemin@ann.jussieu.fr

given function in $\dot H^s(\mathbb{R}^d)$. For a given sequence $(\lambda_n)_{n\in\mathbb{N}}$, the concept of $(\lambda_n)_{n\in\mathbb{N}}$-oscillating sequence is defined. Together with some particular class of Besov spaces, this turns out to be the crucial tool of the proof.

In the third section, we first recall some basic results about incompressible Navier-Stokes system. Then we prove some result about bounded sequences of initial data in the spirit of the work by I. Gallagher (see [14]). And we apply this result to prove the celebrated result by L. Escauriaza, G. Serëgin, V. Sverak (see [12]) which claims that if a solution u to the incompressible three dimensional Navier-Stokes equation develops a singularity at time $T^\star$, then we have

$$\limsup_{t\to T^\star} \|u(t)\|_{\dot H^{\frac12}} = \infty.$$

We follow the approach developed by C. Kenig and G. Koch in [20].

1 Sobolev embeddings revisited

1.1 Sobolev spaces and Sobolev embedding

Definition 1.1. *Let s be a real number. The homogeneous Sobolev space $\dot H^s(\mathbb{R}^d)$ is the set of tempered distributions u the Fourier transform of which $\widehat u$ belongs to $L^1_{loc}(\mathbb{R}^d)$ and satisfies*

$$\|u\|^2_{\dot H^s} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat u(\xi)|^2 d\xi < \infty.$$

The following interpolation inequality is of constant use. We left as an exercise to the reader the proof that $\dot H^s(\mathbb{R}^d)$ is a Hilbert space in the case when s is less than $d/2$.

Proposition 1.1. *If u belongs to $\dot H^{s_1} \cap \dot H^{s_2}$, for any s between s_1 and s_2, u belongs on $\dot H^s$ and*

$$\|u\|_{\dot H^s} \leq \|u\|^{\theta}_{\dot H^{s_1}} \|u\|^{1-\theta}_{\dot H^{s_2}} \quad \textit{with} \quad s = \theta s_1 + (1-\theta) s_2.$$

Proof. Let us simply apply Hölder inequality with $p = \frac{1}{\theta}$ and $q = \frac{1}{1-\theta}$ to the functions $\xi \mapsto |\xi|^{2\theta s_1}$, $\xi \mapsto |\xi|^{2(1-\theta)s_2}$ and the Borel measure $|\widehat u(\xi)|^2 d\xi$ and this gives the result. □

Let us state the classical Sobolev inequality and its dual version.

Theorem 1.1. *If s belongs to $[0, \frac{d}{2}[$, then the space $\dot H^s(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$. If p belongs to $]1,2]$, then the space $L^p(\mathbb{R}^d)$ is continuously included in the space $\dot H^{d\left(\frac12 - \frac1p\right)}$.*

Proof. The second part is easily deduced from the first one proceeding by duality. Let us write that

$$\|a\|_{\dot H^s} = \sup_{\|\varphi\|_{\dot H^{-s}(\mathbb{R}^d)} \leq 1} \langle a, \varphi \rangle.$$

As $s = d\left(\frac{1}{2} - \frac{1}{p}\right) = d\left(1 - \frac{1}{p} - \frac{1}{2}\right)$, we have by the first part

$$\|\varphi\|_{L^{\overline{p}}} \leq C\|\varphi\|_{\dot{H}^{-s}}$$

where $\overline{p}$ is the conjugate of p defined by $\frac{1}{p} + \frac{1}{\overline{p}} = 1$ and thus

$$\begin{aligned}\|a\|_{\dot{H}^s} &\leq C \sup_{\|\varphi\|_{L^{\overline{p}}}\leq 1} \langle a, \varphi\rangle\\ &\leq C\|a\|_{L^p}.\end{aligned}$$

This concludes the proof of the second part. □

There are many different proofs of the first part. We shall use a frequency cut off argument which gives for free a refined version of this inequality which will be crucial in the second chapter. Let us introduce the following definition.

Definition 1.2. *Let θ be a function of $\mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\theta}$ be compactly supported, has value 1 near 0 and satisfies $0 \leq \widehat{\theta} \leq 1$. For u in $\mathcal{S}'(\mathbb{R}^d)$ and $\sigma > 0$, we set*

$$\|u\|_{\dot{B}^{-\sigma}} \stackrel{\text{def}}{=} \sup_{A>0} A^{d-\sigma}\|\theta(A\cdot) \star u\|_{L^\infty}.$$

The fact that $B^{-\sigma}$ is a Banach space is an exercise left to the reader. We shall see later on that the space is independent of the choice of the function θ. Let us observe that if u belongs to $\dot{H}^s$, then $\widehat{u}$ is locally in L^1 and the function $\widehat{\theta}(A^{-1}\cdot)\widehat{u}$ is in L^1. The inverse Fourier theorem implies that

$$\begin{aligned}\|A^d\theta(A\cdot) \star u\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{\theta}(A^{-1}\cdot)\widehat{u}\|_{L^1}\\ &\leq (2\pi)^{-d}\int_{\mathbb{R}^d} \widehat{\theta}(A^{-1}\xi)|\xi|^{-s}|\xi|^s|\widehat{u}(\xi)|\, d\xi.\end{aligned}$$

Using the fact that $\widehat{\theta}$ is compactly supported, Cauchy-Schwarz inequality implies that

$$\|A^d\theta(A\cdot) \star u\|_{L^\infty} \leq \frac{C}{\left(\frac{d}{2} - s\right)^{\frac{1}{2}}} A^{\frac{d}{2}-s}\|u\|_{\dot{H}^s}.$$

This means exactly that the space $\dot{H}^s$ is continuously included in $\dot{B}^{s-\frac{d}{2}}$. By multiplication, we can assume that $\|u\|_{\dot{B}^{s-\frac{d}{2}}} = 1$. Then let us estimate $\|u\|_{L^p}$. We decompose the function u in low and high frequencies. More precisely, let us write

$$u = u_{\ell,A} + u_{h,A} \quad \text{with} \quad u_{\ell,A} = \mathcal{F}^{-1}(\widehat{\theta}(A^{-1}\cdot)\widehat{u}) \tag{1.1}$$

where θ is the function of Definition 1.2. The triangle inequality implies that

$$\big(|u| > \lambda\big) \subset \big(|u_{\ell,A}| > \lambda/2\big) \cup \big(|u_{h,A}| > \lambda/2\big)\cdot$$

By definition of $\|\cdot\|_{\dot{B}^{s-\frac{d}{2}}}$, we have $\|u_{\ell,A}\|_{L^\infty} \leq A^{\frac{d}{2}-s}$. From this, we deduce that

$$A = A_\lambda \stackrel{\text{def}}{=} \left(\frac{\lambda}{2}\right)^{\frac{p}{d}} \Longrightarrow \mu\big(|u_{\ell,A}| > \lambda/2\big) = 0.$$

We deduce that

$$\|u\|_{L^p}^p \leq p \int_0^\infty \lambda^{p-1} \mu(|u_{h,A_\lambda}| > \lambda/2)\, d\lambda.$$

Using that

$$\mu(|u_{h,A_\lambda}| > \lambda/2) \leq 4 \frac{\|u_{h,A_\lambda}\|_{L^2}^2}{\lambda^2},$$

we get

$$\|u\|_{L^p}^p \leq 4p \int_0^\infty \lambda^{p-3} \|u_{h,A_\lambda}\|_{L^2}^2\, d\lambda.$$

Because the Fourier transform is (up to a constant) an isometry on $L^2(\mathbb{R}^d)$ and the function $\widehat{\theta}$ has value 1 near 0, we thus get for some $c > 0$ depending only on $\widehat{\theta}$,

$$\|u\|_{L^p}^p \leq 4p\,(2\pi)^{-d} \int_0^\infty \lambda^{p-3} \int_{(|\xi| \geq cA_\lambda)} |\widehat{u}(\xi)|^2\, d\xi\, d\lambda \tag{1.2}$$

for some positive constant c. Now, by definition of A_λ, we have

$$|\xi| \geq cA_\lambda \iff \lambda \leq C_\xi \stackrel{\text{def}}{=} 2\Big(\frac{|\xi|}{c}\Big)^{\frac{d}{p}}.$$

Fubini's theorem thus implies that

$$\begin{aligned}\|u\|_{L^p}^p &\leq 4p\,(2\pi)^{-d} \int_{\mathbb{R}^d} \left(\int_0^{C_\xi} \lambda^{p-3} d\lambda\right) |\widehat{u}(\xi)|^2\, d\xi \\ &\leq (2\pi)^{-d} \frac{p2^p}{p-2} \int_{\mathbb{R}^d} \Big(\frac{|\xi|}{c}\Big)^{\frac{d(p-2)}{p}} |\widehat{u}(\xi)|^2\, d\xi.\end{aligned}$$

As $s = d\left(\frac{1}{2} - \frac{1}{p}\right)$, the theorem is proved. □

1.2 Interpretation in terms of Besov spaces and oscillations

In fact the above proof tells more than the classical Sobolev theorem, namely the following theorem.

Theorem 1.2. *Let s be in $]0, d/2[$. There exists a constant C depending only on d and $\widehat{\theta}$ such that*

$$\|u\|_{L^p} \leq \frac{C}{(p-2)^{\frac{1}{p}}} \|u\|_{\dot{B}^{s-\frac{d}{2}}}^{1-\frac{2}{p}} \|u\|_{\dot{H}^s}^{\frac{2}{p}} \quad \textit{with} \quad p = \frac{2d}{d-2s}\cdot$$

Let us see what type of improvement it is compared with the classical inequality. Let φ be a given function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and ω a unit vector of $\mathbb{R}^d$. Let us consider the family of functions defined by

$$u_\varepsilon(x) = e^{i\frac{(x|\omega)}{\varepsilon}} \varphi(x).$$

Let us prove that for any σ in $]0, d]$ we have

$$\|u_\varepsilon\|_{\dot{B}^{-\sigma}} \lesssim \varepsilon^\sigma. \tag{1.3}$$

By Hölder's inequality, we have

$$A^d\|\theta(A\cdot) \star \phi_\varepsilon\|_{L^\infty} \leq \|\theta\|_{L^1}\|\phi\|_{L^\infty}.$$

From this we deduce that, if $A\varepsilon \geq 1$ then we have

$$A^{d-\sigma}\|\theta(A\cdot) \star \phi_\varepsilon\|_{L^\infty} \leq \varepsilon^\sigma\|\theta\|_{L^1}\|\phi\|_{L^\infty}. \tag{1.4}$$

If $A\varepsilon \leq 1$, we perform integrations by parts. More precisely, using that

$$(-i\varepsilon\partial_1)^d e^{i\frac{x_1}{\varepsilon}} = e^{i\frac{x_1}{\varepsilon}}$$

and Leibniz formula, we get

$$\begin{aligned} A^d(\theta(A\cdot) \star \phi_\varepsilon)(x) &= (iA\varepsilon)^d \int_{\mathbb{R}^d} \partial_{y_1}^d \left(\theta(A(x-y))\phi(y)\right) e^{i\frac{y_1}{\varepsilon}}\, dy \\ &= (iA\varepsilon)^d \sum_{k\leq d} \binom{d}{k} A^k((-\partial_1)^k\theta)(A\cdot) \star (e^{i\frac{y_1}{\varepsilon}}\partial_1^{d-k}\phi)(x). \end{aligned}$$

Using Hölder's inequalities, we get that

$$A^k \left\|((-\partial_1)^k\theta)(A\cdot) \star (e^{i\frac{y_1}{\varepsilon}}\partial_1^{d-k}\phi)\right\|_{L^\infty} \leq \|\partial_1^k\theta\|_{L^{\frac{d}{k}}}\|\partial_1^{d-k}\phi\|_{L^{(\frac{d}{k})'}}.$$

Thus, we get $A^d\|\theta(A\cdot) \star \phi_\varepsilon\|_{L^\infty} \leq C(A\varepsilon)^d$. As we are in the case when $A\varepsilon \leq 1$, we get, for any $\sigma \leq d$,

$$A^d\|\theta(A\cdot) \star \phi_\varepsilon\|_{L^\infty} \leq C(A\varepsilon)^\sigma.$$

Together with (1.4), this concludes the proof of Inequality (1.3).

Considering that $\|u_\varepsilon\|_{\dot{H}^s} \lesssim \varepsilon^{-s}$, then we can check that

$$\|u_\varepsilon\|_{\dot{B}^{s-\frac{d}{2}}}^{1-\frac{2}{p}}\|u_\varepsilon\|_{\dot{H}^s}^{\frac{2}{p}} \lesssim \varepsilon^{(\frac{d}{2}-s)(1-\frac{2}{p})-\frac{2s}{p}} \sim 1.$$

This shows that the refined inequality of Theorem 1.2 is invariant under translation in Fourier spaces (i.e. multiplication by oscillating functions).

The spaces defined in Definition 1.2 have a universal property: they are the biggest normed spaces which are translation invariant and which have the same scaling. More precisely we have the following proposition.

Proposition 1.2. *Let E be a norm space continuously included in the space of tempered distribution. Let us assume that the space E is globally invariant under dilations and translations and that a constant C and a positive real number σ exists such that*

$$\|u(\lambda \cdot - \overrightarrow{a})\|_E \leq C\lambda^{-\sigma}\|u\|_E.$$

Then the space E is continuously embedded in $B^{-\sigma}$.

Proof. As E is continuously included in $\mathcal{S}'$, then we have

$$|\langle u, \theta\rangle| \leq C\|u\|_E.$$

Because of the hypothesis on E, we get

$$|\langle u(A^{-1}\cdot +x), \theta\rangle| \leq \|u(A^{-1}\cdot +x)\|_E \leq CA^\sigma\|u\|_E.$$

As we have

$$\langle u(A^{-1}\cdot +x), \theta\rangle = A^d(\theta(A\cdot) \star u)(x),$$

we get the result. □

1.3 The link with Besov norms

The following proposition is an exercise about partition of unity.

Proposition 1.3. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}$. There exist two radial functions χ and φ valued in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,4/3))$ and to $\mathcal{D}(\mathcal{C})$, and such that*

$$\forall \xi \in \mathbb{R}^d,\ \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \tag{1.5}$$

$$\forall \xi \in \mathbb{R}^d\setminus\{0\},\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{1.6}$$

$$|j-j'| \geq 2 \Rightarrow \operatorname{Supp}\ \varphi(2^{-j}\cdot) \cap \operatorname{Supp}\ \varphi(2^{-j'}\cdot) = \emptyset, \tag{1.7}$$

$$j \geq 1 \Rightarrow \operatorname{Supp}\ \chi \cap \operatorname{Supp}\ \varphi(2^{-j}\cdot) = \emptyset, \tag{1.8}$$

the set $\widetilde{\mathcal{C}} \stackrel{\text{def}}{=} B(0,2/3) + \mathcal{C}$ is an annulus and we have

$$|j-j'| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^j\mathcal{C} = \emptyset. \tag{1.9}$$

Besides, we have

$$\forall \xi \in \mathbb{R}^d,\ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{1.10}$$

$$\forall \xi \in \mathbb{R}^d\setminus\{0\},\ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{1.11}$$

Let us state the following definition.

Definition 1.3. *Let s be a real number, and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of distributions u of $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|^r_{L^p}\right)^{\frac{1}{r}} < \infty.$$