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Series on Analysis, Applications and Computation – Vol. 6

M W Wong

An Introduction to Pseudo-Differential Operators

3rd Edition

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藏书章

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Vol. 6: An Introduction to Pseudo-Differential Operators, 3rd Edition
by M W Wong

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Preface

There have been a lot of developments in pseudo-differential operators since the first edition was published in 1991. The second edition, published in 1999, has served well as an introduction to pseudo-differential operators in the capacity of a textbook. The prerequisites for the first two editions are minimal and can be seen from the prefaces for the first and second editions, which follow this one.

The third edition is intended to contain not only improvements of some of the contents and additional exercises to some of the existing chapters in the second edition, but also new chapters to make the book more useful without losing the original intent of keeping the book as elementary as possible. The new chapters notwithstanding, the whole book remains to be a textbook primarily for beginning graduate students in mathematics. It is also useful to mathematicians aspiring to do research in pseudo-differential operators and related topics.

The new chapters are the last seven chapters, Chapters 17–23, of the book. The focus of Chapters 17–21 is on the class of pseudo-differential operators studied in the first two editions. The theme underlying Chapters 17–19 is Gårding's inequality, which is used to prove the existence and uniqueness of solutions of pseudo-differential equations. In particular, the Hille–Yosida–Phillips theorem on one-parameter semigroups is used to prove the existence and uniqueness of solutions of initial value problems for heat equations governed by pseudo-differential operators. After a chapter, Chapter 20, on the general theory of Fredholm operators that we need for this book, the ellipticity and Fredholmness of pseudo-differential operators are developed in Chapter 21. The ellipticity, Fredholmness, an index formula and the spectral invariance for another class of pseudo-differential operators, dubbed symmetrically global pseudo-differential operators in this book, are studied in Chapters 22 and 23. The emphasis of the book, as in the first two editions, is on the global theory of elliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

As we are now well into the new millennium and moving forward with increasing acceleration, many advanced topics in any area of science and engineering in 1991 are now being taught in basic courses to students. The prerequisites for a complete understanding of the book can be succinctly described as a first course in functional analysis including the Riesz theory of compact operators. The book contains ample material to be studied leisurely and carefully for a two-semester course. One-semester courses can be designed by omitting certain topics in order to fulfil the needs of the students and the duration of the semester.

Preface to the Second Edition

The first edition of the book has been used as the textbook for the standard graduate course in partial differential equations at York University since its publication in 1991. The motivation for writing the second edition stems from the desire to remove several deficiencies and obscurities, and to incorporate the improvements that I can see through many years of teaching the subject to graduate students and discussions of the subject with colleagues. Notwithstanding the many changes I have in mind, I am convinced that the elementary character of the book has served and will serve well as an ideal introduction to the study of pseudo-differential operators. Thus, the basic tenet of the second edition is to retain the style and the scope of the first edition.

Notable in the second edition is the addition of two chapters to the book. Experience in teaching pseudo-differential operators reveals the fact that many graduate students are still not comfortable with the interchange of order of integration and differentiation. The new chapter added to the beginning of the book is to prove a theorem to this effect which can cope with every interchange of order of integration and differentiation encountered in the book. Another new chapter, added as the final chapter in the second edition, is to prove a theorem on the existence of weak solutions of pseudo-differential equations. The inclusion of this chapter, in my opinion, enhances the value of the book as a book on partial differential equations. Furthermore, it provides a valuable connection with the chapter on minimal and maximal operators and the chapter on global regularity.

Other new features in the second edition include a deeper study of elliptic operators and parametrices, more details on the proof of the L^p-boundedness of pseudo-differential operators, additional exercises in several chapters of the book, a slightly expanded bibliography and an index.

Preface to the First Edition

The aim of the book is to give a straightforward account of a class of pseudo-differential operators. The prerequisite for understanding the book is a course in real variables. It is hoped that the book can be used in courses in functional analysis, Fourier analysis and partial differential equations.

The first eight chapters of the book contain the basic formal calculus of pseudo-differential operators. The remaining five chapters are devoted to some topics of a more functional analytic character.

It is clear to the expert that the book takes up a single theme in a wide subject and many important topics are omitted. It is my belief that this approach is in fact a more effective introduction of pseudo-differential operators to mathematicians and graduate students beginning to learn the subject. Exercises are included in the text. They are useful to anyone who wants to understand and appreciate the book better.

The actual writing of the book was essentially carried out and completed at the University of California at Irvine while I was on sabbatical leave from York University in the academic year in 1987–88. The preliminary drafts of the book have been used in seminars and graduate courses at the University of California at Irvine and York University.

Many colleagues and students have helped me improve the contents and organization of the book. In particular, I wish to thank Professor William Margulies at the California State University at Long Beach, Professor Martin Schechter at the University of California at Irvine, Professor Tuan Vu and Mr. Zhengbin Wang at York University for their stimulating conversations and critical comments about my book. I also wish to thank Mr. Lian Pi, my Ph.D. research student at York University, who has worked out every exercise in the book.

Contents

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Chapter 1

Introduction, Notation and Preliminaries

Let $\mathbb{R}^n$ be the usual Euclidean space given by

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_j\text{'s are real numbers}\}.$$

We denote points in $\mathbb{R}^n$ by x, y, ξ, η etc. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be any two points in $\mathbb{R}^n$. The inner product $x \cdot y$ of x and y is defined by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j$$

and the norm $|x|$ of x is defined by

$$|x| = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2}.$$

On $\mathbb{R}^n$, the simplest differential operators are $\frac{\partial}{\partial x_j}$, $j = 1, 2, \ldots, n$. We sometimes denote $\frac{\partial}{\partial x_j}$ by ∂_j. For reasons we shall see later in the book, we usually find the operator D_j given by $D_j = -i\partial_j$, $i^2 = -1$, better in expressing certain formulas.

The most general linear partial differential operator of order m on $\mathbb{R}^n$ treated in this book is of the form

$$\sum_{\alpha_1+\alpha_2+\cdots+\alpha_n \le m} a_{\alpha_1,\alpha_2,\ldots,\alpha_n}(x) D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}, \tag{1.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonnegative integers and $a_{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$ is an infinitely differentiable complex-valued function on $\mathbb{R}^n$. To simplify the expression (1.1), we let

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),$$

$$|\alpha| = \sum_{j=1}^{n} \alpha_j$$

and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}.$$

The α, given by an n-tuple of nonnegative integers, is called a *multi-index.* We call $|x|$ the *length* of the multi-index α. With the help of multi-indices, we can rewrite our differential operator (1.1) in the better form

$$\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha. \tag{1.2}$$

For each fixed x in $\mathbb{R}^n$, the operator (1.2) is a polynomial in $D_1, D_2, \ldots, D_n$. Therefore it is natural to denote the operator (1.2) by $P(x, D)$. If we replace D in (1.2) by a point $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ in $\mathbb{R}^n$, then we obtain a polynomial $\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ in $\mathbb{R}^n$, where $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. Naturally, this polynomial is denoted by $P(x, \xi)$. We call $P(x, \xi)$ the *symbol* of the operator $P(x, D)$.

In this book, we shall study the partial differential operators (1.2) and their generalizations called pseudo-differential operators. To do this, we find it convenient to introduce in Chapters 2–5 certain aspects of analysis pertinent to our need.

The following list of remarks, notation and formulas will be useful to us.

(i) We denote the set of all real numbers by $\mathbb{R}$ and the set of all complex numbers by $\mathbb{C}$.

(ii) All vector spaces are assumed to be over the field of complex numbers. All functions are assumed to be complex-valued unless otherwise specified.

(iii) We do not bother to distinguish a function f from its value $f(x)$ at x. In other words, we shall occasionally use the symbol $f(x)$ to denote the function f without any warning.

(iv) Although the differential operator $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ is more useful to us, we still use the differential operator $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ very often in the book. In case we want to emphasize the variable x (or ξ) with respect to which we differentiate, we write ∂_x^α (or ∂_ξ^α) for ∂^α and D_x^α (or D_ξ^α) for D^α.

(v) We denote the set of all infinitely differentiable functions on $\mathbb{R}^n$ by $C^\infty(\mathbb{R}^n)$.

(vi) The L^p norm of a function f in $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, is denoted by $\|f\|_p$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be any two multi-indices.

(vii) $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for $j = 1, 2, \ldots, n$.

(viii) $\alpha - \beta$ is the multi-index $(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_n - \beta_n)$ whenever $\beta \le \alpha$.

(ix) $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$.

(x) $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n}$ whenever $\beta \le \alpha$.

(xi) The formula

$$D^\alpha(fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^\beta f)(D^{\alpha-\beta} g) \tag{1.3}$$

is known as Leibniz's formula. It is a special case of the following more general Leibniz's formula.

(xii) Let $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ be a linear partial differential operator with constant coefficients, and $P(\xi)$ its symbol. Then

$$P(D)(fg) = \sum_{|\mu| \le m} \frac{1}{\mu!} (P^{(\mu)}(D)f)(D^\mu g),$$

where $P^{(\mu)}(D)$ is the linear partial differential operator with symbol $P^{(\mu)}(\xi)$ given by

$$P^{(\mu)}(\xi) = (\partial^\mu P)(\xi), \quad \xi \in \mathbb{R}^n.$$

(xiii) Let $f \in C^\infty(\mathbb{R}^n)$. Then

$$D^\alpha \left(\frac{1}{f}\right) = \sum C_{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}} \frac{(\partial^{\alpha^{(1)}} f)(\partial^{\alpha^{(2)}} f) \cdots (\partial^{\alpha^{(k)}} f)}{f^{k+1}}, \tag{1.4}$$

where $C_{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}}$'s are constants and the sum is taken over all possible multi-indices $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}$, which form a partition of α. The formula (1.4) is valid at all points x in $\mathbb{R}^n$ for which $f(x) \ne 0$.

(xiv) Let f be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Then for $1 \le p < \infty$,

$$\left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x,y)\, dy \right|^p dx \right\}^{1/p} \le \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |f(x,y)|^p dx \right\}^{1/p} dy. \tag{1.5}$$

This inequality is the well-known *Minkowski's inequality in integral form.*

(xv) The inequality

$$|x^\alpha| \le |x|^{|\alpha|} \tag{1.6}$$

for all $x \in \mathbb{R}^n$ and multi-indices α, which will be used quite often in this book, is an inequality in terms of the absolute value of a real number, the norm of a point in $\mathbb{R}^n$ and the length of a multi-index. Its proof is left as an exercise.

Exercises

1.1. Find the symbol of each of the following partial differential operators on $\mathbb{R}^2$.

(i) $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$

(ii) $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$

(iii) $\frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_2^2}$

(iv) $\frac{\partial}{\partial x_1} + i\frac{\partial^2}{\partial x_2^2}$

(v) $\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$

1.2. For each of the partial differential operators in Exercise 1.1, find the zero set of the symbol, i.e., the set of zeros of the symbol.

1.3. Find the symbol of the partial differential operator

$$P(x, D) = \frac{\partial^2}{\partial x_1^2} + x_1^2 \frac{\partial^2}{\partial x_2^2}$$

on $\mathbb{R}^2$. For each fixed $x \in \mathbb{R}^2$, find the zero set $\{\xi \in \mathbb{R}^2 : P(x, \xi) = 0\}$.

1.4. What is the analog of Minkowski's inequality in (1.5) when $p = \infty$?

1.5. Prove inequality (1.6).

Chapter 2

Differentiation of Integrals Depending on Parameters

The aim of this chapter is to prove a theorem on how to differentiate an integral depending on parameters in order to justify every interchange of integration and differentiation throughout the book. I hope analysts can find the theorem, or some variant of it, useful. Other criteria can be found on, e.g., p.288 in [Friedman (1971)], pp.94–95 in [Royden (1988)], and p.85 in [Wheeden and Zygmund (1977)].

Theorem 2.1. *Let* (Y, μ) *be a measure space and* $f : \mathbb{R}^n \times Y \to \mathbb{C}$ *be a measurable function such that*
(i) $f(x, \cdot) \in L^1(Y)$ *for all* x *in* $\mathbb{R}^n$,
(ii) $f(\cdot, y) \in C^\infty(\mathbb{R}^n)$ *for almost all* y *in* Y,
(iii) $\sup_{x\in\mathbb{R}^n} \int_Y |(\partial_x^\alpha f)(x, y)| d\mu < \infty$ *for all multi-indices* α.
Then the integral $\int_Y f(x, y)\, d\mu$*, as a function of* x*, is in* $C^\infty(\mathbb{R}^n)$ *and*

$$\partial^\beta \int_Y f(x, y)\, d\mu = \int_Y (\partial_x^\beta f)(x, y)\, d\mu, \quad x \in \mathbb{R}^n,$$

for all multi-indices β.

We begin with a lemma.

Lemma 2.2. *Let* $f : \mathbb{R}^n \times Y \to \mathbb{C}$ *be such that the hypotheses of Theorem 2.1 are satisfied. Then for any multi-index* α *and* $j = 1, 2, \ldots, n$*, the integrals*

$$\int_Y (\partial_x^\alpha f)(x_1, \ldots, x_j, \ldots, x_n, y)\, d\mu$$

and

$$\int_Y |(\partial_x^\alpha f)(x_1, \ldots, x_j, \ldots, x_n, y)|\, d\mu,$$

as functions of x_j*, are continuous on* $\mathbb{R}$.

Proof Using the mean value theorem and hypothesis (iii), we obtain

$$\int_Y \{|(\partial_x^\alpha f)(x_1,\ldots,x_j+h,\ldots,x_n,y)| - |(\partial_x^\alpha f)(x_1,\ldots,x_j,\ldots,x_n,y)|\}\,d\mu \le M_{\alpha,j}|h|$$

for all x in $\mathbb{R}^n$, where

$$M_{\alpha,j} = \sup_{x\in\mathbb{R}^n} \int_Y |(\partial_x^\alpha \partial_{x_j} f)(x,y)|\,d\mu,$$

and hence the lemma. □

Proof of Theorem 2.1 In view of of the proof of Lemma 2.2, the theorem is valid for the zero multi-index. Suppose that the theorem is valid for any multi-index with length l and let γ be a multi-index with length $l+1$. If we write γ as $\beta+\varepsilon$, where β is a multi-index with length l, and ε is a multi-index with length one and the only nonzero entry in the jth position, then, by the fundamental theorem of calculus, Lemma 2.2 and the Fubini theorem,

$$\begin{aligned}
&\partial^\gamma \int_Y f(x,y)\,d\mu \\
&= \partial_j \int_Y (\partial_x^\beta f)(x,y)\,d\mu \\
&= \lim_{h\to 0}\frac{1}{h}\int_Y \{(\partial_x^\beta f)(x_1,\ldots,x_j+h,\ldots,x_n,y) - (\partial_x^\beta f)(x_1,\ldots,x_j,\ldots,x_n,y)\}\,d\mu \\
&= \lim_{h\to 0}\frac{1}{h}\int_Y \left\{\int_{x_j}^{x_j+h} (\partial_x^\gamma f)(x_1,\ldots,s,\ldots,x_n,y)\,ds\right\} d\mu \\
&= \lim_{h\to 0}\frac{1}{h}\int_{x_j}^{x_j+h} \left\{\int_Y (\partial_x^\gamma f)(x_1,\ldots,s,\ldots,x_n,y)\,d\mu\right\} ds \\
&= \int_Y (\partial_x^\gamma f)(x,y)\,d\mu, \quad x\in\mathbb{R}^n.
\end{aligned}$$

Thus, by induction, the proof is complete. □

From the proof of Theorem 2.1, we obtain the following result.

Corollary 2.3. *The conclusions of Theorem* 2.1 *remain valid if hypothesis* (iii) *is replaced by the hypothesis that, for every multi-index* α, *the integral* $\int_Y (\partial_x^\gamma f)(x,y)\,d\mu$, *as a function of* x, *is continuous on* $\mathbb{R}^n$.

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