DALE HUSEMOLLER

FIBRE BUNDLES

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PREFACE

The notion of a fibre bundle first arose out of questions posed in the 1930s on the topology and geometry of manifolds. By the year 1950 the definition of fibre bundle had been clearly formulated, the homotopy classification of fibre bundles achieved, and the theory of characteristic classes of fibre bundles developed by several mathematicians, Chern, Pontrjagin, Stiefel, and Whitney. Steenrod's book, which appeared in 1950, gave a coherent treatment of the subject up to that time.

About 1955 Milnor gave a construction of a universal fibre bundle for any topological group. This construction is also included in Part I along with an elementary proof that the bundle is universal.

During the five years from 1950 to 1955, Hirzebruch clarified the notion of characteristic class and used it to prove a general Riemann-Roch theorem for algebraic varieties. This was published in his Ergebnisse Monograph. A systematic development of characteristic classes and their applications to manifolds is given in Part III and is based on the approach of Hirzebruch as modified by Grothendieck.

In the early 1960s, following lines of thought in the work of A. Grothendieck, Atiyah and Hirzebruch developed K-theory, which is a generalized cohomology theory defined by using stability classes of vector bundles. The Bott periodicity theorem was interpreted as a theorem in K-theory, and J. F. Adams was able to solve the vector field problem for spheres, using K-theory. In Part II an introduction to K-theory is presented, the nonexistence of elements of Hopf invariant 1 proved (after a proof of Atiyah), and the proof of the vector field problem sketched.

I wish to express gratitude to S. Eilenberg, who gave me so much encouragement during recent years, and to J. C. Moore, who read parts of

the manuscript and made many useful comments. Conversations with J. F. Adams, R. Bott, A. Dold, and F. Hirzebruch helped to sharpen many parts of the manuscript. During the writing of this book, I was particularly influenced by the Princeton notes of J. Milnor and the lectures of F. Hirzebruch at the 1963 Summer Institute of the American Mathematical Society.

Dale Husemoller

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1
PRELIMINARIES ON HOMOTOPY THEORY

In this introductory chapter, we consider those aspects of homotopy theory that will be used in later sections of the book. This is done in outline form. References to the literature are included.

Two books on homotopy theory, those by Hu [1]† and Hilton [1], contain much of the background material for this book. In particular, chapters 1 to 5 of Hu [1] form a good introduction to the homotopy needed in fibre bundle theory.

1. CATEGORY THEORY AND HOMOTOPY THEORY

A homotopy $f_t\colon X \to Y$ is a continuous one-parameter family of maps, and two maps f and g are homotopically equivalent provided there is a homotopy f_t with $f = f_0$ and $g = f_1$. Since this is an equivalence relation, one can speak of a homotopy class of maps between two spaces.

As with the language of set theory, we use the language of category theory throughout this book. For a good introduction to category theory, see MacLane [1].

We shall speak of the category **sp** of (topological) spaces, (continuous) maps, and composition of maps. The category **H** of spaces, homotopy classes of maps, and composition of homotopy classes is a quotient category. Similarly, we speak of maps and of homotopy classes of maps that preserve base points. The associated categories of pointed spaces (i.e., spaces with base points) are denoted $\mathbf{sp}_0$ and $\mathbf{H}_0$, respectively.

The following concept arises frequently in fibre bundle theory.

1.1 Definition. Let X be a set, and let Φ be a family of spaces M whose underlying sets are subsets of X. The Φ-topology on X is defined by requiring a set U in X to be open if and only if $U \cap M$ is open in M for each $M \in \Phi$. If X is a space and if Φ is a family of subspaces of X, the topology on X is said to be Φ-defined provided the Φ-topology on the set X is the given topology on X.

† Bracketed numbers refer to bibliographic entries at end of book.

For example, if X is a Hausdorff space and if Φ is a family of compact subspaces, X is called a k-space if the topology of X is Φ-defined. If $M_1 \subset M_2 \subset \cdots \subset X$ is a sequence of spaces in a set X, the inductive topology on X is the Φ-topology, where $\Phi = \{M_1, M_2, \ldots\}$.

The following are examples of unions of spaces which are given the inductive topology.

$$\mathbf{R}^1 \subset \mathbf{R}^2 \subset \cdots \subset \mathbf{R}^n \subset \cdots \subset \mathbf{R}^\infty = \bigcup_{1 \leq n} \mathbf{R}^n$$

$$\mathbf{C}^1 \subset \mathbf{C}^2 \subset \cdots \subset \mathbf{C}^n \subset \cdots \subset \mathbf{C}^\infty = \bigcup_{1 \leq n} \mathbf{C}^n$$

$$S^1 \subset S^2 \subset \cdots \subset S^n \subset \cdots \subset S^\infty = \bigcup_{1 \leq n} \mathbf{C}^n$$

$$RP^1 \subset RP^2 \subset \cdots \subset RP^n \subset \cdots \subset RP^\infty = \bigcup_{1 \leq n} RP^n$$

$$CP^1 \subset CP^2 \subset \cdots \subset CP^n \subset \cdots \subset CP^\infty = \bigcup_{1 \leq n} CP^n$$

Above, RP^n denotes the real projective space of lines in $\mathbf{R}^{n+1}$, and CP^n denotes the complex projective space of complex lines in $\mathbf{C}^{n+1}$. We can view RP^n as the quotient of S^n with x and $-x$ identified, and we can view CP^n as the quotient of $S^{2n+1} \subset \mathbf{C}^{n+1}$, where the circle $ze^{i\theta}$ for $0 \leqq \theta \leqq 2\pi$ is identified to a point.

It is easily proved that each locally compact space is a k-space. The spaces S^∞, RP^∞, and CP^∞ are k-spaces that are not locally compact.

2. COMPLEXES

The question of whether or not a map defined on a subspace prolongs to a larger subspace frequently arises in fibre bundle theory. If the spaces involved are CW-complexes and the subspaces are subcomplexes, a satisfactory solution of the problem is possible.

A good introduction to this theory is the original paper of J. H. C. Whitehead [1, secs. 4 and 5]. Occasionally, we use relative cell complexes (X,A), where A is a closed subset of X and $X - A$ is a disjoint union of open cells with attaching maps. The reader can easily generalize the results of Whitehead [1] to relative cell complexes. In particular, one can speak of relative CW-complexes. If X^n is the n-skeleton of a CW-complex, then (X,X^n) is a relative CW-complex.

The prolongation theorems for maps defined on CW-complexes follow from the next proposition.

2.1 Proposition. Let (X,A) be a relative CW-complex having one cell C with an attaching map $u_C: I^n \to X = A \cup C$, and let $f: A \to Y$ be a map. Then f extends to a map $g: X \to Y$ if and only if $fu_C: \partial I^n \to Y$ is null homotopic.

A space Y is said to be connected in dimension n provided every map $S^{n-1} \to Y$ is null homotopic or, in other words, prolongs to a map $B^n \to Y$. From (2.1) we easily get the following result.

2.2 Theorem. Let (X,A) be a relative CW-complex, and let Y be a space that is connected in each dimension for which X has cells. Then each map $A \to X$ prolongs to a map $X \to Y$.

As a corollary of (2.2), a space is contractible, i.e., homotopically equivalent to a point, if and only if it is connected in each dimension.

The above methods yield the result that the homotopy extension property holds for CW-complexes; see Hilton [1, p. 97].

The following theorems are useful in considering vector bundles over CW-complexes. Since they do not seem to be in the literature, we give details of the proofs.

If C is a cell in a CW-complex X and if $u_C: B^n \to X$ is the attaching map, then $u_C(0)$ is called the center of C.

2.3 Theorem. Let (X,A) be a finite-dimensional CW-complex. Then there exists an open subset V of X with $A \subset V \subset X$ such that A is a strong deformation retract of V with a homotopy h_t. This can be done so that V contains the center of no cell C of X, and if U_A is an open subset of A, there is an open subset U_X of X with $U_X \cap A = U_A$ and $h_t(U_X) \subset U_X$ for $t \in I$.

Proof. We prove this theorem by induction on the dimension of X. For dim $X = -1$, the result is clear. For $X^n = X$, let V' be an open subset of X_{n-1} with $A \subset V' \subset X^{n-1}$ and a contracting homotopy $h'_t: V' \to V'$. Let U' be the open subset of V' with $U' \cap A = U_A$ and $h_t(U') \subset U'$ for $t \in I$. This is given by the inductive hypothesis.

For each n-cell C, let $u_C: B^n \to X$ be the attaching map of C, and let V'_C denote the open subset $u_C^{-1}(V')$ of ∂B^n and U'_C denote $u_C^{-1}(U')$. Let M_C denote the closed subset of all ty for $t \in [0, 1]$ and $y \in \partial B^n - V'_C$. There is an open subset V of X with $V \cap X_{n-1} = V'$ and $u_C^{-1}(V) = B^n - M_C$, that is, $y \in u_C^{-1}(V)$ if and only if $y \neq 0$ and $y/\|y\| \in V'_C$, and there is an open subset U_X of V with $U_X \cap X^{n-1} = U'$ and $y \in u_C^{-1}(U_X)$ if and only if $y \neq 0$ and $y/\|y\| \in U'_C$.

We define a contracting homotopy $h_t: V \to V$ by the following requirements: $h_t(u_C(y)) = u_C(2ty/\|y\| + (1 - 2t)y)$ for $y \in B^n$, $t \in [0, \frac{1}{2}]$, $h_t(x) = x$ for $x \in V'$, $t \in [0, \frac{1}{2}]$, $h_t(x) = h'_{2t-1}(h_{1/2}(x))$ for $t \in [\frac{1}{2}, 1]$, where h'_t is defined in the first paragraph. Then A is a strong deformation retract of V, and $h_t(U_X) \subset U_X$ by the character of the radial construction. Finally, we have $u_C(0) \notin V$ for each cell C of X. This proves the theorem.

2.4 Remark. With the notation of Theorem (2.3), if U_A is contractible, U_X is contractible.

2.5 Theorem. Let X be a finite CW-complex with m cells. Then X can be covered by m contractible open sets.

Proof. We use induction on m. For $m = 1$, X is a point, and the statement is clearly true. Let C be a cell of maximal dimension. Then X equals a subcomplex A of $m - 1$ cells with C attached by a map u_C. There are $V'_1, \ldots, V'_{m-1}$ contractible open sets in A which prolong by (2.3) and (2.4) to contractible open sets $V_1, \ldots, V_{m-1}$ of X which cover A. If V_m denotes $C = u_C(\operatorname{int} B^n)$, then $V_1, \ldots, V_m$ forms an open contractible covering of X.

2.6 Theorem. Let X be a CW-complex of dimension n. Then X can be covered by $n + 1$ open sets $V_0, \ldots, V_n$ such that each path component of V_i is contractible.

Proof. For $n = 0$ the statement of the theorem clearly holds, and we use induction on n. Let $V'_0, \ldots, V'_n$ be an open covering of the $(n - 1)$-skeleton of X, where each component of V'_i is a contractible set. Let V be an open neighborhood of X^{n-1} in X with a contracting homotopy leaving X^{n-1} elementwise fixed $h_t\colon V \to V$ onto X^{n-1}. Using (2.3), we associate with each component of V'_i an open contractible set in V. The union of these disjoint sets is defined to be V_i. Let V_n be the union of the open n cells of X. The path components of V_n are the open n cells. Then the open covering $V_0, \ldots, V_n$ has the desired properties.

3. THE SPACES MAP (X,Y) AND MAP$_0$ (X,Y)

For two spaces X and Y, the set Map (X,Y) of all maps $X \to Y$ has several natural topologies. For our purposes the compact-open topology is the most useful. If $\langle K,V \rangle$ denotes the subset of all $f \in$ Map (X,Y) with $f(K) \subset V$ for $K \subset X$ and $V \subset Y$, the compact-open topology is generated by all sets $\langle K,V \rangle$ such that K is a compact subset of X and V is an open subset of Y.

The subset Map$_0$ (X,Y) of base point preserving maps is given the subspace topology.

The spaces Map (X,Y) are useful for homotopy theory because of the natural map

$$\theta\colon \text{Map}\ (Z \times X,\ Y) \to \text{Map}\ (Z,\ \text{Map}\ (X,Y))$$

which assigns to $f(z,x)$ the map $Z \to$ Map (X,Y), where the image of $z \in Z$ is the map $x \mapsto f(z,x)$. This map

$$\text{Map}\ (Z \times X,\ Y) \to \text{Map}\ (Z,\ \text{Map}\ (X,Y))$$

is a homeomorphism onto its image set for Hausdorff spaces. Moreover, we have the following proposition by an easy proof.

3.1 Proposition. For two spaces X and Y, the function

$$\theta\colon \text{Map}\ (Z \times X,\ Y) \to \text{Map}\ (Z,\ \text{Map}\ (X,Y))$$

is bijective if and only if the substitution function $\sigma\colon \text{Map}\ (X,Y) \times X \to Y$, where $\sigma(f,x) = f(x)$, is continuous.

The substitution function $\sigma\colon \text{Map}\ (X,Y) \times X \to Y$ is continuous for X locally compact. By applying (3.1) to the case $Z = I$, the closed unit interval, we see that a homotopy from X to Y, that is, a map $X \times I \to Y$, can be viewed as a path in Map (X,Y).

A map similar to θ can be defined for base point preserving maps defined on compact spaces X and Z, using the reduced product $Z \wedge X = (Z \times X)/(Z \vee X)$. Here $Z \vee X$ denotes the disjoint union of Z and X with base points identified. The space $Z \vee X$ is also called the wedge product. The map corresponding to θ is defined:

$$\text{Map}_0\ (Z \wedge X,\ Y) \to \text{Map}_0\ (Z,\ \text{Map}_0\ (X,Y))$$

It is a homeomorphism for Z and X compact spaces or for Z and X two CW-complexes.

Let 0 be the base point of $I = [0, 1]$, and view S^1 as $[0, 1]/\{0, 1\}$. The following functors $\mathbf{sp_0} \to \mathbf{sp_0}$ are very useful in homotopy theory.

3.2 Definition. The cone over X, denoted $C(X)$, is $X \wedge I$; the suspension of X, denoted $S(X)$, is $X \wedge S^1$; the path space of X, denoted $P(X)$, is $\text{Map}_0\ (I,X)$; and the loop space of X, denoted $\Omega(X)$, is $\text{Map}_0\ (S^1,X)$.

A point of $C(X)$ or $S(X)$ is a class $\langle x,t \rangle$ determined by a pair $(x,t) \in X \times I$, where $\langle x_0,t \rangle = \langle x,0 \rangle =$ base point of $C(X)$ or $S(X)$ and, in addition, $\langle x,1 \rangle =$ base point of $S(X)$. If $f\colon X \to Y$ is a map, $C(f)(\langle x,t \rangle) = \langle f(x),t \rangle$ defines a map $C(f)\colon C(X) \to C(Y)$, and $S(f)(\langle x,t \rangle) = \langle f(x),t \rangle$ defines a map $S(f)\colon S(X) \to S(Y)$; with these definitions, $C\colon \mathbf{sp_0} \to \mathbf{sp_0}$ and $S\colon \mathbf{sp_0} \to \mathbf{sp_0}$ are functors. Also, we consider the map $\omega\colon X \to C(X)$, where $\omega(x) = \langle x,1 \rangle$. Then $S(X)$ equals $C(X)/\omega(X)$. Since S^1 is $[0,1]$ with its two end points pinched to a point, one can easily check that the equal sets $S(X)$ and $C(X)/\omega(X)$ have the same topologies.

Path space $P(X)$ can be viewed as the subspace of paths $u\colon I \to X$ such that $u(0) = x_0$, and $\Omega(X)$ as the subspace of paths $u\colon I \to X$ such that $u(0) = u(1) = x_0$. If $f\colon X \to Y$ is a map, then $P(f)u = fu$ defines a map $P(f)\colon P(X) \to P(Y)$, and $\Omega(f)u = fu$ defines a map $\Omega(f)\colon \Omega(X) \to \Omega(Y)$. With these definitions, $P\colon \mathbf{sp_0} \to \mathbf{sp_0}$ and $\Omega\colon \mathbf{sp_0} \to \mathbf{sp_0}$

are functors. Also, we consider the map $\pi\colon P(X) \to X$, where $\pi(u) = u(1)$. Then $\Omega(X)$ equals $\pi^{-1}(x_0)$ as a subspace.

3.3 Proposition. The functions $\omega\colon 1_{\mathbf{sp}_0} \to C$ and $\pi\colon P \to 1_{\mathbf{sp}_0}$ are morphisms of functors.

Proof. If $f\colon X \to Y$ is a map, then $\omega(f(x)) = \langle f(x), 1\rangle = C(f)\omega(x)$ for each $x \in X$, and $f\pi(u) = fu(1) = \pi(P(f)u)$ for each $u \in P(X)$.

3.4 Proposition. The following statements are equivalent for a base point preserving map $f\colon X \to Y$.

(1) The map f is homotopic to the constant.
(2) There exists a map $g\colon C(X) \to Y$ with $g\omega = f$.
(3) There exists a map $h\colon X \to P(Y)$ with $\pi h = f$.

Proof. Condition (1) says that there is a map $f^*\colon X \times I \to Y$ with $f^*(x, 0) = y_0$, $f^*(x, 1) = f(x)$, and $f^*(x_0, t) = y_0$. The existence of f^* is equivalent to the existence of $g\colon C(X) \to Y$, where $g\langle x,1\rangle = f(x)$. The existence of f^* is equivalent to the existence of $h\colon X \to P(Y)$, where $h(x)(1) = f(x)$.

3.5 Proposition. The spaces $C(X)$ and $P(X)$ are contractible.
Proof. Let $h_s\colon C(X) \to C(X)$ be the homotopy defined by $h_s(\langle x,t\rangle) = \langle x,st\rangle$. Then h_1 is the identity, and h_0 is constant. Similarly, let $k_s\colon P(X) \to P(X)$ be the homotopy defined by $k_s(u)(t) = u(st)$.

As an easy application of Proposition (3.1), we have the next theorem.

3.6 Theorem. There exists a natural bijection $\alpha\colon [S(X),Y]_0 \to [X, \Omega(Y)]_0$, where $\alpha[f\langle x,t\rangle] = [(\theta f)(x)(t)]$.

4. HOMOTOPY GROUPS OF SPACES

Let $[X,Y]_0$ denote base point preserving homotopy classes of maps $X \to Y$. A multiplication on a pointed space Y is a map $\phi\colon Y \times Y \to Y$. The map θ defines a function $\phi_X\colon [X,Y]_0 \times [X,Y]_0 \to [X,Y]_0$ for each space X, by composition. If $([X,Y]_0;\phi_X)$ is a group for each X, then (Y,ϕ) is called a homotopy associative H-space. The loop space ΩY is an example of a homotopy associative H-space, where $\phi\colon \Omega Y \times \Omega Y \to \Omega Y$ is given by the following relation:

$$\phi(u,v)(t) = \begin{cases} u(2t) & \text{for} \quad 0 \leqq t \leqq \frac{1}{2} \\ v(2t-1) & \text{for} \quad \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

A comultiplication on a pointed space X is a map $\psi\colon X \to X \vee X$. The map ψ defines a function $\psi^Y\colon [X,Y]_0 \times [X,Y]_0 \to [X,Y]_0$ for each space Y, by composition. If $([X,Y]_0, \psi^Y)$ is a group for each Y, then (X,ψ)