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Selected Topics in Finite Element Methods

Zhiming Chen and Haijun Wu

(有限元方法选讲)

SCIENCE PRESS
Beijing

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Preface to the Series in Information and Computational Science

Since the 1970s, Science Press has published more than thirty volumes in its series Monographs in Computational Methods. This series was established and led by the late academician, Feng Kang, the founding director of the Computing Center of the Chinese Academy of Sciences. The monograph series has provided timely information of the frontier directions and latest research results in computational mathematics. It has had great impact on young scientists and the entire research community, and has played a very important role in the development of computational mathematics in China.

To cope with these new scientific developments, the Ministry of Education of the People's Republic of China in 1998 combined several subjects, such as computational mathematics, numerical algorithms, information science, and operations research and optimal control, into a new discipline called Information and Computational Science. As a result, Science Press also reorganized the editorial board of the monograph series and changed its name to Series in Information and Computational Science. The first editorial board meeting was held in Beijing in September 2004, and it discussed the new objectives, and the directions and contents of the new monograph series.

The aim of the new series is to present the state of the art in Information and Computational Science to senior undergraduate and graduate students, as well as to scientists working in these fields. Hence, the series will provide concrete and systematic expositions of the advances in information and computational science, encompassing also related interdisciplinary developments.

I would like to thank the previous editorial board members and assistants, and all the mathematicians who have contributed significantly to the monograph series on Computational Methods. As a result of their contributions the monograph series achieved an outstanding reputation in the community. I sincerely wish that we will extend this support to the new Series in Information and Computational Science, so that the new series can equally enhance the scientific development in information and computational science in this century.

Shi Zhongci

2005.7

Preface

This book grows out of the lectures the first author gave in the summer of 2002 in the Institute of Computational Mathematics of Chinese Academy of Sciences. The purpose of the lectures was to present a concise introduction to the basic ideas and mathematical tools in the construction and analysis of finite element methods for solving partial differential equations so that the students can start to do research on the theory and applications of the finite element method after the summer course. Some of the materials of the book have been taught several times by the authors in Nanjing University and Peking University. The current form of the book is based on the lecture notes which are constantly updated and expanded reflecting the newest development of the topics through the years.

The finite element method is nowadays penetrating into almost every aspect of the scientific and engineering applications. There have already been several excellent monographs on the mathematical theory and applications of finite element methods. In this book we do not present the results in the most general form but rather in a form that we believe the extension to the general form would be possible with minor efforts. The topics covered in the book reflect our own research experiences and so many of the important aspects of finite element methods such as nonconforming finite element methods and domain decomposition methods are not touched. Nevertheless we hope the ideas and methods introduced in the book will also be useful for the readers who are interested in other topics of finite element methods.

The book consists of two parts. The first part includes the first six chapters and the last chapter which present the fundamental theory of finite element methods for solving the second order elliptic equations. In this part we consider the variational formulation of partial differential equations including Sobolev spaces, the construction of finite element methods, the a priori error analysis of finite element methods, the a posteriori error estimation and adaptive finite element methods, the multigrid method, and the mixed finite element methods. The last chapter is devoted to the implementation of the finite element method based on the MATLAB PDE Toolbox. The second part of the book introduces three topics of active current research. We consider adaptive finite element methods for parabolic equations in Chapter 7 and finite element methods for time-harmonic Maxwell equations in Chapter 8. We introduce multiscale finite element methods in Chapter 9 which play an increasingly important role in the simulation of flow transport in heterogeneous porous media. The MATLAB codes in Chapter 10 can be downloaded at the webpage http://lsec.cc.ac.cn/~zmchen/codes.html.

There exists a huge literature on the finite element methods. The references of the book are far from being complete. At the end of each chapter we add some notes on the references which are directly related to the material covered in the book. Many of the important references are not included in the book. The readers are encouraged to consult the monographs and the references cited in the book for historical and further information.

The prerequisite of the book is the standard undergraduate course on functional analysis and differential equations of mathematical physics. The results required in the book on Sobolev spaces are summarized in Chapter 1 without proof. Our experience shows that it is usually sufficient for the readers to understand the theory of finite element methods without first knowing the proofs of the Sobolev embedding theorem and the trace theorem. Except Chapter 1 the book is largely self-contained. In very few places, the readers are referred to the literature for results that are outside the scope of the book.

Finally we would like to thank our colleagues and collaborators for their insights and discussions through the years. We wish to express our thanks to Ms. Xianhua Meng for typing the manuscript. We also thank the Science Press for their efforts to make the publishing of the book possible.

Zhiming Chen
Haijun Wu
March 5, 2010

Contents

Chapter 1

Variational Formulation of Elliptic Problems

In this chapter we shall introduce the variational formulation of the elliptic boundary value problem

$$Lu = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^d$ ($d = 1, 2, 3$) and $u : \Omega \to \mathbb{R}$ is the unknown function. Here $f : \Omega \to \mathbb{R}$ is a given function and L denotes the second-order partial differential operator of the form

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u \tag{1.2}$$

for given coefficients a_{ij}, b_i, c, $i, j = 1, 2, \cdots, d$.

We shall assume the partial differential operator L is *uniformly elliptic*, that is, there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^d.$$

1.1 Basic concepts of Sobolev space

Let Ω be an open subset in $\mathbb{R}^d$. We define $C_0^\infty(\Omega)$ to be the linear space of infinitely differentiable functions with compact support in Ω. Let $L^1_{\text{loc}}(\Omega)$ be the set of locally integrable functions:

$$L^1_{\text{loc}}(\Omega) = \left\{f : \ f \in L^1(K), \quad \forall \text{ compact set } K \subset \text{ interior } \Omega\right\}.$$

We start with the definition of weak derivatives.

Definition 1.1 Assume $f \in L^1_{\text{loc}}(\Omega), 1 \leqslant i \leqslant d$, we say $g_i \in L^1_{\text{loc}}(\Omega)$ is the *weak partial derivative* of f with respect to x_i in Ω if

$$\int_\Omega f\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega g_i\varphi\,\mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega).$$

We write

$$\partial_{x_i} f = \frac{\partial f}{\partial x_i} = g_i, \quad i = 1, 2, \cdots, d, \quad \nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d} \right)^{\mathrm{T}}.$$

Similarly, for a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$, α_j non-negative integers, with length $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\partial^\alpha f \in L^1_{\text{loc}}(\Omega)$ is defined by

$$\int_\Omega \partial^\alpha f \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi \, \mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega),$$

where $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}$.

Example 1.1 *Let $d = 1, \Omega = (-1,1)$, and $f(x) = 1 - |x|$. The weak derivative of f is*

$$g = \begin{cases} 1, & \text{if } x \leqslant 0, \\ -1, & \text{if } x > 0. \end{cases}$$

The weak derivative of g does not exist.

Definition 1.2 (Sobolev space) For a non-negative integer k and a real $p \geqslant 1$, we define

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), \text{ for all } |\alpha| \leqslant k\}.$$

The space is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum\limits_{|\alpha| \leqslant k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}, & 1 \leqslant p < +\infty, \\ \max\limits_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$. It is also a Banach space. When $p = 2$, we denote

$$H^k(\Omega) = W^{k,2}(\Omega), \quad H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

The space $H^k(\Omega)$ is a Hilbert space when equipped with the inner product

$$(u, v)_{k,\Omega} = \sum_{|\alpha| \leqslant k} \int_\Omega \partial^\alpha u \partial^\alpha v \, \mathrm{d}x.$$

Example 1.2 (1) *Let $\Omega = (0, 1)$ and consider the function $u = x^\alpha$. One easily verifies that $u \in L^2(\Omega)$ if $\alpha > -1/2$, $u \in H^1(\Omega)$ if $\alpha > 1/2$, and $u \in H^k(\Omega)$ if $\alpha > k - 1/2$.*

(2) *Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1/2\}$ and consider the function $f(x) = \ln \big| \ln |x| \big|$. Then $f \in W^{1,p}(\Omega)$ for $p \leqslant 2$ but $f \notin L^\infty(\Omega)$. This example shows that functions in $H^1(\Omega)$ are neither necessarily continuous nor bounded.*

Now we consider the regularization of functions in Sobolev space. Let ρ be a non-negative, real-valued function in $C_0^\infty(\mathbb{R}^d)$ with the property

$$\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1, \quad \operatorname{supp}(\rho) \subset \{x : |x| \leqslant 1\}. \tag{1.3}$$

An example of such a function is

$$\rho(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geqslant 1, \end{cases} \tag{1.4}$$

where the constant C is so chosen that $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$. For $\epsilon > 0$, the function $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$ belongs to $C_0^\infty(\mathbb{R}^d)$ and $\operatorname{supp}(\rho_\epsilon) \subset \{x : |x| < \epsilon\}$. ρ_ϵ is called the *mollifier* and the convolution

$$u_\epsilon(x) = (\rho_\epsilon * u)(x) = \int_{\mathbb{R}^d} \rho_\epsilon(x-y)u(y)\,\mathrm{d}y \tag{1.5}$$

is called the *regularization* of u. Regularization has several important and useful properties that are summarized in the following lemma.

Lemma 1.1 (i) *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then for every $\epsilon > 0$, $u_\epsilon \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(\rho_\epsilon * u) = (\partial^\alpha \rho_\epsilon) * u$ for each multi-index α;*

(ii) *If $u \in C(\mathbb{R}^d)$, then u_ε converges uniformly to u on compact subsets of $\mathbb{R}^d$;*

(iii) *If $u \in L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, then $u_\epsilon \in L^p(\mathbb{R}^d)$, $\|u_\epsilon\|_{L^p(\mathbb{R}^d)} \leqslant \|u\|_{L^p(\mathbb{R}^d)}$, and $\lim\limits_{\epsilon\to 0} \|u_\epsilon - u\|_{L^p(\mathbb{R}^d)} = 0$.*

Proof (i) follows directly from (1.5).

(ii) is obvious by observing that

$$\begin{aligned} |u_\varepsilon(x) - u(x)| &\leqslant \int_{\mathbb{R}^d} \rho_\epsilon(x-y)\,|u(x)-u(y)|\,\mathrm{d}y \\ &\leqslant (\max\rho)\varepsilon^{-d} \int_{|y-x|\leqslant\varepsilon} |u(x)-u(y)|\,\mathrm{d}y \end{aligned}$$

and that u is uniformly continuous on compact sets.

To show (iii), let $p' \in \mathbb{R}$ such that $1/p + 1/p' = 1$. Then by Hölder inequality

$$\begin{aligned} \|u_\epsilon\|_{L^p(\mathbb{R}^d)} &\leqslant \left\{ \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |u(y)|\rho_\epsilon(x-y)\,\mathrm{d}y \right)^p \mathrm{d}x \right\}^{1/p} \\ &\leqslant \left\{ \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |u(y)|^p \rho_\epsilon(x-y)\,\mathrm{d}y \right) \cdot \left(\int_{\mathbb{R}^d} \rho_\epsilon(x-y)\,\mathrm{d}y \right)^{p/p'} \mathrm{d}x \right\}^{1/p} \\ &= \left\{ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(y)|^p \rho_\epsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \right\}^{1/p} \\ &= \|u\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{1.6}$$

For $u \in L^p(\mathbb{R}^d)$ and any $\delta > 0$, we choose a continuous function v with compact support such that $\|u - v\|_{L^p(\mathbb{R}^d)} \leqslant \delta/3$. From (ii), $\|v_\epsilon - v\|_{L^p(\mathbb{R}^d)} \leqslant \delta/3$ for ϵ sufficiently small. By the triangle inequality and (1.6),

$$\|u_\epsilon - u\|_{L^p(\mathbb{R}^d)} \leqslant \|u_\epsilon - v_\epsilon\|_{L^p(\mathbb{R}^d)} + \|v_\epsilon - v\|_{L^p(\mathbb{R}^d)} + \|u - v\|_{L^p(\mathbb{R}^d)} \leqslant \delta. \quad (1.7)$$

This completes the proof of (iii). □

In this book, a *domain* is referred to an open and connected set. The following lemma will be useful in proving the Poincaré-Friedrichs inequality.

Lemma 1.2 *Let Ω be a domain, $u \in W^{1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, and $\nabla u = 0$ a.e. on Ω, then u is constant on Ω.*

Proof For any bounded subdomain K of Ω and $\epsilon > 0$, let K_ϵ be the ϵ-neighborhood of K, that is, K_ϵ is the union of all balls $B(x, \epsilon)$, $x \in K$. Let u be extended to be zero outside Ω and let $u_\epsilon = u * \rho_\epsilon$. If $K_\epsilon \subset \Omega$ for some $\epsilon > 0$, then $\nabla u_\epsilon = (\nabla u) * \rho_\epsilon = 0$ in K. Since u_ϵ is smooth, we deduce that u_ϵ is constant in K. On the other hand, by Lemma 1.1, $u_\epsilon \to u$ in $L^1(K)$. Thus u is constant in K. This completes the proof. □

Theorem 1.1 (Properties of weak derivatives) *Assume* $1 \leqslant p < +\infty$.

(i) (*Product rule*) *If f,$g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $fg \in W^{1,p}(\Omega)$ and*

$$\frac{\partial (fg)}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \quad \text{a.e. } in \ \Omega, \quad i = 1,2,\cdots,d;$$

(ii) (*Chain rule*) *If $f \in W^{1,p}(\Omega)$ and $F \in C^1(\mathbb{R}), F' \in L^\infty(\mathbb{R})$, then $F(f) \in W^{1,p}(\Omega)$ and*

$$\frac{\partial F(f)}{\partial x_i} = F'(f) \frac{\partial f}{\partial x_i} \quad \text{a.e. } in \ \Omega, \quad i = 1,2,\cdots,d;$$

(iii) *If $f \in W^{1,p}(\Omega)$ and F is piecewise smooth on $\mathbb{R}$ with $F' \in L^\infty(\mathbb{R})$, then $F(f) \in W^{1,p}(\Omega)$. Furthermore, if $\mathcal{L}$ is the set of all corner points of F, we have, for $i = 1,2,\cdots,d$,*

$$\frac{\partial F(f)}{\partial x_i} = \begin{cases} F'(f) \dfrac{\partial f}{\partial x_i}, & if \ f(x) \notin \mathcal{L}, \\ 0, & if \ f(x) \in \mathcal{L}. \end{cases}$$

In order to introduce further properties of Sobolev spaces, we introduce the following condition on the boundary of the domain.

Definition 1.3 (Lipschitz domain) We say that a domain Ω has a *Lipschitz boundary* $\partial\Omega$ if for each point $x \in \partial\Omega$ there exist $r > 0$ and a Lipschitz mapping $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that — upon rotating and relabeling the coordinate axes if necessary — we have

$$\Omega \cap Q(x,r) = \{y : \varphi(y_1, \cdots, y_{d-1}) < y_d\} \cap Q(x,r),$$

where $Q(x,r) = \{y : |y_i - x_i| < r, i = 1, 2, \cdots, d\}$(see Figure 1.1). We call Ω a Lipschitz domain if it has a Lipschitz boundary.

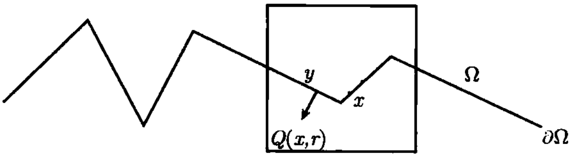

Figure 1.1 The domain with a Lipschitz boundary

Theorem 1.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^d$.*

(i) *Let $\mathcal{D}(\bar{\Omega})$ be the set of all functions $\varphi|_\Omega, \varphi \in C_0^\infty(\mathbb{R}^d)$. Then $\mathcal{D}(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$ for all integers $k \geqslant 0$ and real p with $1 \leqslant p < \infty$;*

(ii) *Let $u \in W^{k,p}(\Omega)$ and let $\widetilde{u}$ denote its extension by zero outside Ω. If $\widetilde{u} \in W^{k,p}(\mathbb{R}^d)$, for $k \geqslant 1, 1 \leqslant p < \infty$, then $u \in W_0^{k,p}(\Omega)$;*

(iii) *If in addition Ω is bounded and $k \geqslant 1, 1 \leqslant p \leqslant \infty$, there exists a continuous linear extension operator $\mathbb{E}$ from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^d)$ such that $\mathbb{E}u = u$ in Ω.*

The following theorem plays an important role in the application of Sobolev spaces.

Theorem 1.3 (Sobolev Imbedding Theorem) *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 \leqslant p \leqslant \infty$. Then*

(i) *If $0 \leqslant k < d/p$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^q(\Omega)$ with $q = dp/(d - kp)$ and compactly imbedded in $L^{q'}(\Omega)$ for any $1 \leqslant q' < q$;*

(ii) *If $k = d/p$, the space $W^{k,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for any $1 \leqslant q < \infty$;*

(iii) *If $0 \leqslant m < k - d/p < m + 1$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $C^{m,\alpha}(\bar{\Omega})$ for $\alpha = k - d/p - m$, and compactly imbedded in $C^{m,\beta}(\bar{\Omega})$ for all $0 \leqslant \beta < \alpha$.*

Example 1.3 *$H^1(\Omega)$ is continuously imbedded in $C^{0,1/2}(\bar{\Omega})$ for $d = 1$, in $L^q(\Omega)$, $1 \leqslant q < \infty$, for $d = 2$, and in $L^6(\Omega)$ for $d = 3$.*

Theorem 1.4 (Poincaré-Friedrichs Inequality) *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 \leqslant p \leqslant \infty$. Then*

$$\|u\|_{L^p(\Omega)} \leqslant C_p \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega),$$
$$\|u - u_\Omega\|_{L^p(\Omega)} \leqslant C_p \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega),$$

where $u_\Omega = \dfrac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$.

Proof We only give the proof of the first inequality. Assume it is false. Then there exists a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} = 1, \quad \|\nabla u_n\|_{L^p(\Omega)} \leqslant \frac{1}{n}.$$

By the compactness imbedding theorem, there exist a subsequence (still denoted by) u_n and a function $u \in L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$. By the completeness of $L^p(\Omega)$ we know that $\nabla u_n \to 0$ in $L^p(\Omega)^d$. Thus, by the definition of weak derivative, $\nabla u = 0$, which implies, by Lemma 1.2, that $u = 0$. This contradicts the fact that $\|u\|_{L^p(\Omega)} = 1$. □

Next we study the trace of functions in $W^{k,p}$ for which we first introduce the Sobolev spaces of non-integer order k. There are several definitions of fractional Sobolev spaces which unfortunately are not equivalent. Here we shall use the following one.

Definition 1.4 (Fractional Sobolev space) For two real numbers s, p with $p \geqslant 1$ and $s = k + \sigma$ where $\sigma \in (0, 1)$. We define $W^{s,p}(\Omega)$ when $p < \infty$ as the set of all functions $u \in W^{k,p}(\Omega)$ such that

$$\int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{d+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty, \quad \forall |\alpha| = k.$$

Likewise, when $p = \infty$, $W^{s,\infty}(\Omega)$ is the set of all functions $u \in W^{k,\infty}(\Omega)$ such that

$$\max_{|\alpha|=k} \operatorname*{ess\,sup}_{x,y\in\Omega, x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\sigma} < \infty, \quad \forall |\alpha| = k.$$

$W^{s,p}(\Omega)$ when $p < \infty$ is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left\{ \|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{d+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \right\}^{1/p}$$

with the obvious modification when $p = \infty$.

The closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$ is denoted by $W_0^{s,p}(\Omega)$. It is also a Banach space. When $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^s(\Omega) = W_0^{s,2}(\Omega)$.

We remark that the statement of Sobolev Imbedding Theorem (Theorem 1.3) is valid for fractional Sobolev spaces. The density result and the extension result in Theorem 1.2 are valid as well for fractional Sobolev spaces when $s > 0$.

Now we examine the boundary values of functions in $W^{s,p}(\Omega)$. The fractional Sobolev space $W^{s,p}(\Gamma)$ on the boundary Γ of Ω can be defined by using the atlas of the boundary Γ and using the definition of fractional Sobolev space in Definition 1.3 locally. As we are mostly interested in the case when $s < 1$ we make use of the following equivalent definition of Sobolev space on the boundary.

Definition 1.5 (Sobolev space on the boundary) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with boundary Γ. Let s, p be two real numbers with $0 \leqslant s < 1$ and $1 \leqslant p < \infty$. We define $W^{s,p}(\Gamma)$ as the set of all functions $u \in L^p(\Omega)$ such that

$$\int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{d-1+sp}} \, \mathrm{d}s(x) \, \mathrm{d}s(y) < \infty.$$

$W^{s,p}(\Gamma)$ is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Gamma)} = \left\{ \|u\|^p_{L^p(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{d-1+sp}} \, \mathrm{d}s(x) \, \mathrm{d}s(y) \right\}^{1/p}.$$

As usual, when $p = 2$, $H^s(\Gamma) = W^{s,2}(\Gamma)$.

We know that if u is continuous on $\bar{\Omega}$ then its restriction to the boundary $\partial\Omega$ is well-defined and continuous. If however, u is a function in some Sobolev space, the restriction $u|_{\partial\Omega}$ may not be defined in a pointwise sense. To interpret boundary values of Sobolev functions properly, we introduce the following trace theorem for Sobolev spaces.

Theorem 1.5 (Trace Theorem) *Let Ω be a bounded Lipschitz domain with boundary Γ, $1 \leqslant p < \infty$, and $1/p < s \leqslant 1$.*

(i) *There exists a bounded linear mapping*

$$\gamma_0 : W^{s,p}(\Omega) \text{ onto } W^{s-1/p,p}(\Gamma)$$

such that $\gamma_0(u) = u$ on Γ for all $u \in W^{s,p}(\Omega) \cap C(\bar{\Omega})$;

(ii) *For all $v \in C^1(\bar{\Omega})$ and $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega u \frac{\partial v}{\partial x_i} \, \mathrm{d}x = - \int_\Omega \frac{\partial u}{\partial x_i} v \, \mathrm{d}x + \int_\Gamma \gamma_0(u) \, v \, n_i \, \mathrm{d}s,$$

where n_i denotes the i-th component of the unit outward normal to Γ;

(iii) $W^{1,p}_0(\Omega) = \{u \in W^{1,p}(\Omega) : \gamma_0(u) = 0\}$;

(iv) γ_0 *has a continuous right inverse, that is, there exists a constant C such that, $\forall g \in W^{s-1/p,p}(\Gamma)$, there exists $u_g \in W^{s,p}(\Omega)$ satisfying*

$$\gamma_0(u_g) = g \quad \text{and} \quad \|u_g\|_{W^{s,p}(\Omega)} \leqslant C \|g\|_{W^{s-1/p,p}(\Gamma)}.$$

$\gamma_0(u)$ is called the trace of u on the boundary $\Gamma = \partial\Omega$. Noting that γ_0 is surjective and the property (iv) is a consequence of (i) and the open mapping theorem. The function u_g is said to be a *lifting* of g in $W^{s,p}(\Omega)$. In what follows, whenever no confusion can arise, we write u instead of $\gamma_0(u)$ on boundaries.

1.2 Variational formulation

We assume $f \in L^2(\Omega)$ and the coefficients in (1.2) satisfy $a_{ij}, b_i, c \in L^\infty(\Omega), i, j = 1, 2, \cdots, d$.

Assuming for the moment the solution u is a smooth function, we multiply $Lu = f$ in (1.1) by a smooth function $\varphi \in C_0^\infty(\Omega)$, and integrate over Ω, to find

$$\int_\Omega \left(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} \varphi + cu\varphi \right) \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x, \tag{1.8}$$

where we have used the integration by parts formula in Theorem 1.5 in the first term on the left hand side. There are no boundary terms since $\varphi = 0$ on $\partial\Omega$. By the density argument we deduce that (1.8) is valid for any $\varphi \in H_0^1(\Omega)$, and the resulting equation makes sense if $u \in H_0^1(\Omega)$. We choose the space $H_0^1(\Omega)$ to incorporate the boundary condition from (1.1) that "$u = 0$" on $\partial\Omega$. This motivates us to define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ as follows

$$a(u, \varphi) = \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} \varphi + cu\varphi \Big) \mathrm{d}x.$$

Definition 1.6 $u \in H_0^1(\Omega)$ is called a *weak solution* of the boundary value problem (1.1) if

$$a(u, \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1(\Omega),$$

where $(\cdot, \cdot)$ denotes the inner product on $L^2(\Omega)$.

More generally, we can consider the boundary value problem (1.1) for $f \in H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$. For example, f is defined by

$$\langle f, \varphi \rangle = \int_\Omega \Big(f_0 \varphi + \sum_{i=1}^d f_i \frac{\partial \varphi}{\partial x_i} \Big) \mathrm{d}x, \quad \forall \varphi \in H_0^1(\Omega),$$

where $f_i \in L^2(\Omega), i = 0, 1, \cdots, d$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Definition 1.7 Suppose $f \in H^{-1}(\Omega)$. $u \in H_0^1(\Omega)$ is called a weak solution of (1.1) if

$$a(u, \varphi) = \langle f, \varphi \rangle, \quad \forall \varphi \in H_0^1(\Omega).$$