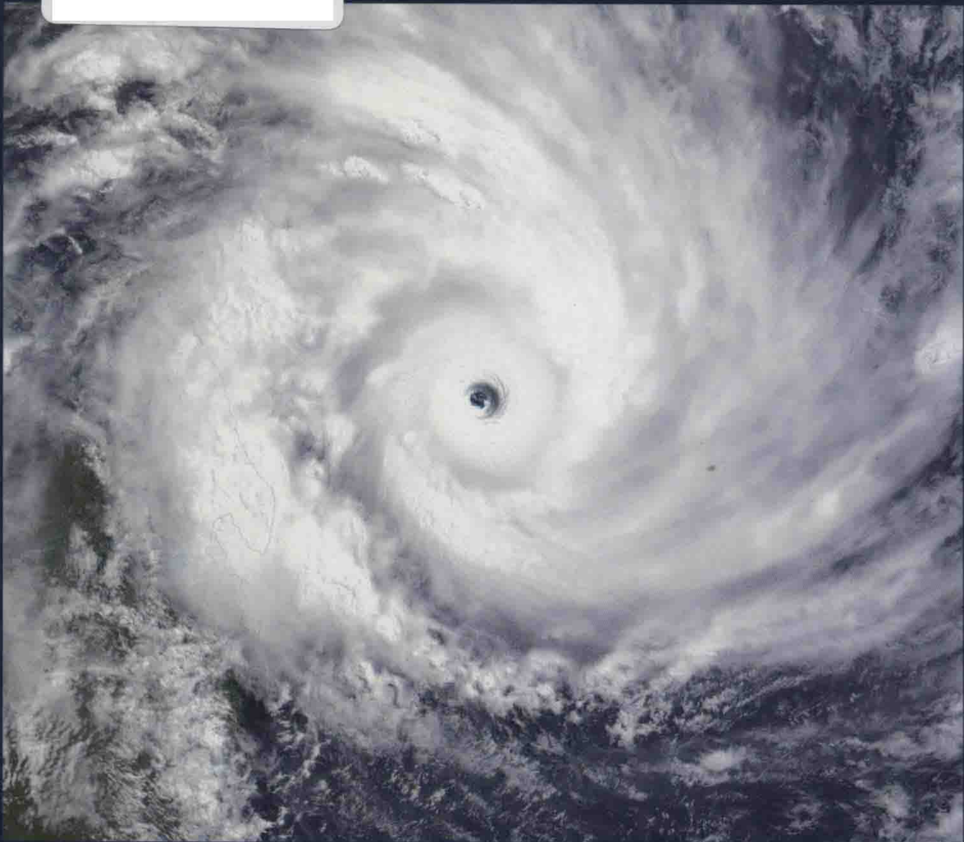
FIXED POINT THEOREMS AND THEIR APPLICATIONS



Ioannis Farmakis • Martin Moskowitz

World Scientific

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Ioannis Farmakis
Martin Moskowitz

City University of New York, USA

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Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Cover: Image courtesy of Jeff Schmaltz, MODIS Land Rapid Response Team at NASA GSFC.

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ISBN 978-981-4458-91-7

Printed in Singapore by World Scientific Printers.

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Preface and Acknowledgments

Our intention here is to show the importance, usefulness and pervasiveness of fixed point theorems in mathematics generally and to try to do so by as elementary and self contained means as possible. The book consists of eight chapters.

Chapter 1, *Early fixed point theorems*, deals mostly with matters connected to the Brouwer fixed point theorem and vector fields on spheres, the Contraction Mapping Principle and affine mappings of finite dimensional spaces. As we shall see, the Brouwer theorem can be generalized in basically two ways, one leading to the Lefschetz fixed point theorem in topology (whose proof is independent of Brouwer) and the other to the Schaüder-Tychonoff fixed point theorem of analysis (which depends on the Brouwer theorem). We shall draw a number of consequences of Brouwer's theorem: among them Perron's theorem concerning eigenvalues of a "positive" operator, which itself has interesting applications to the modern world of computing such as in Google addresses and ranking in professional tennis.

Chapter 2, *Fixed point theorems in Analysis*, deals with the theorems of Schaüder-Tychonoff, as well as the those of F. Hahn, Kakutani and Kakutani-Markov involving groups of affine mappings, the latter two leading naturally to a discussion of Amenable groups.

Chapter 3 concerns *Fixed point theorems in Topology*, particularly the Lefschetz fixed point theorem, the H. Hopf index theorem and some of their consequences. As a further application we prove the conjugacy theorem for maximal tori in a compact connected Lie group. In addition, we explain the Atiyah-Bott fixed point theorem and its relationship to the classical Lefschetz theorem.

Chapter 4, *Fixed point theorems in Geometry*, is devoted first to the fixed point theorem of E. Cartan on compact groups of isometries of Hadamard manifolds and then to fixed point theorems for compact manifolds, when the curvature is negative, due to Preissmann, and positive, due to Weinstein.

Chapter 5 concerns *Fixed points of maps preserving a volume form*, which are important in dynamical systems. We begin with the Poincaré recurrence theorem and deal with some of its philosophical implications. Then we turn to symplectic geometry and fixed point theorems of symplectomorphisms. Here we discuss Arnold's conjecture, prove Poincaré's last geometric theorem and derive a classical result concerning billiards. We then turn to hyperbolic automorphisms of a compact manifold, particularly of a torus, and then to Anosov diffeomorphisms and the analogous Lie algebra automorphisms and explain their significance in dynamics. We conclude this chapter with the Lefschetz zeta function and its many applications.

Chapter 6 deals with the *Fixed point theorem of A. Borel* for a solvable algebraic group acting on a complex projective variety, the most important such varieties being the Grassmann and flag varieties. We then present some consequences of these ideas. A final section concerns a fixed point theorem giving rise to a conjugacy theorem for unipotent subgroups of real reductive linear Lie groups.

We conclude with two brief chapters. Chapter 7 deals with some connections of fixed points to number theory, group theory and complex analysis, while in Chapter 8, *A fixed point theorem in Set Theory*, we prove Tarski's fixed point theorem and, as a consequence, the Schröder-Cantor-Bernstein theorem.

As the reader can see, fixed point theorems are to be found throughout mathematics. These chapters can, for the most part, be read independently. Thus the reader has many options to follow his or her particular interests.

We thank Oleg Farmakis for his help in creating the diagrams and Hossein Abbaspour for reading an earlier draft of the manuscript and making a number of useful suggestions for improvements. Of course, any mistakes are the responsibility of the authors. Finally, we thank Konstantina and Anita for their patience.

New York, January 2013
Ioannis Farmakis, Martin Moskowitz

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Introduction

As exactly a century has passed since the Brouwer fixed point theorem [22] was proved and because in some sense this result is the progenerator of some of the others, it seems appropriate to consider fixed point theorems afresh and in general. As we shall see, these are both quite diverse and pervasive in mathematics. Fixed point theorems are to be found in algebra, analysis, geometry, topology, dynamics, number theory, group theory and even set theory. Before proceeding it would be well to make precise what we mean by a fixed point theorem.

Definition 0.1.1. Let X be a set and $f : X \to X$ be a map from X to itself. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$.

A fixed point theorem is a statement that specifies conditions on X and f guaranteeing that f has a fixed point in X. More generally, we shall sometimes want to consider a family $\mathcal{F}$ of self maps of X. In this context $\mathcal{F}$ is usually a group (or sometimes even a semigroup, see [82]). In this case a fixed point theorem is a statement that specifies conditions on X and $\mathcal{F}$ guaranteeing that there is a *simultaneous* fixed point, $x \in X$, for each $f \in \mathcal{F}$.

When $\mathcal{F}$ is a group G this will usually arise from a *group action* $\phi : G \times X \to X$ of G on X. We will write $\phi(g, x) := gx$. We shall assume that the 1 of G acts as the identity map of X and for all $g, h \in G$ and $x \in X$ that $(gh)x = g(hx)$. We shall call such an X, a *G-space.* Particularly, when X is a topological space and G is a topological group, we shall assume ϕ is *jointly continuous.*

Now, as above, when we have an action of a group G on Y, by taking for "X" the power set, $\mathcal{P}(Y)$, this induces an action of G on $\mathcal{P}(Y)$ and the fixed points of this new action are then precisely the G-invariant subsets of Y. In this way, in many contexts, fixed point theorems give rise to G-invariant sets.

We conclude this introduction with the remark (and example) that the existence of Haar measure μ on a compact topological group G can be regarded as a fixed point theorem. This is because in the associated action of G by left translation on the space $M^+(G)$ of all finite positive measures on G, left invariance simply means μ is G-fixed. Moreover, since μ is finite and can be normalized so that $\mu(G) = 1$, we can even regard it as a G-fixed point under the action of G on the *convex* (and compact in the weak* topology) set of positive normalized measures on G (see Corollary 2.3.5).

Chapter 1

Early Fixed Point Theorems

1.1 The Picard-Banach Theorem

One of the earliest and best known fixed point theorems is that of Picard-Banach 1.1.1. Either explicitly or implicitly this theorem is the usual way one proves *local* existence and uniqueness theorems for systems of ordinary differential equations (see for example [79], sections 7.3 and 7.5). This theorem also can be used to prove the inverse function theorem (see [79], pp. 179-181).

Theorem 1.1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping, that is, one in which there is a $1 > b > 0$ so that for all $x, y \in X$, $d(f(x), f(y)) \leq bd(x, y)$. Then f has a unique fixed point.*

Proof. Choose a point $x_1 \in X$ (in an arbitrary manner) and construct the sequence $x_n \in X$ by $x_{n+1} = f^n(x_1)$, $n \geq 2$. Then x_n is a Cauchy sequence. For $n \geq m$,

$$d(x_n, x_m) = d(f^n(x_1), f^m(x_1)) \leq b^m d(f^{n-m}(x_1), x_1).$$

But

$$d(f^{n-m}(x_1), x_1) \leq d(f^{n-m}(x_1), f^{n-m-1}(x_1)) + \ldots + d(f(x_1), x_1).$$

The latter term is $\leq (b^{n-m-1} + \ldots + b + 1)d(f(x_1), x_1)$ which is itself $\leq \sum_{n=0}^{\infty} b^n d(f(x_1), x_1)$. Since $0 < b < 1$ this geometric series converges to $\frac{1}{1-b} d(f(x_1), x_1)$. Since $b^m \to 0$ we see for n and m sufficiently large, given $\epsilon > 0$,

$$d(x_n, x_m) \leq b^m \frac{1}{1-b} d(f(x_1), x_1) < \epsilon.$$

Hence x_n is Cauchy. Because X is complete $x_n \to x$ for some $x \in X$. As f is a contraction map it is (uniformly) continuous. Hence $f(x_n) \to f(x)$. But as a subsequence $f(x_n) \to x$ so the uniqueness of limits tells us $x = f(x)$.

Now suppose there was another fixed point $y \in X$. Then $d(f(x), f(y)) = d(x, y) \leq bd(x, y)$ so that if $d(x, y) \neq 0$ we conclude $b \geq 1$, a contradiction. Therefore $d(x, y) = 0$ and $x = y$. □

We remark that the reader may wish to consult Bessaga ([11]), or Jachymski, ([60]) where the following converse to the Picard-Banach theorem has been proved.

Theorem 1.1.2. *Let $f : X \to X$ be a self map of a set X and $0 < b < 1$. If f^n has at most 1 fixed point for every integer n, then there exists a metric d on X for which $d(f(x), f(y)) \leq bd(x, y)$, for all x and $y \in X$. If, in addition, some f^n has a fixed point, then d can be chosen to be complete.*

As a corollary to the Picard-Banach theorem we have the following precursor to the Brouwer theorem.

Corollary 1.1.3. *Let X be the closed unit ball in $\mathbb{R}^n$ and $f : X \to X$ be a nonexpanding map; that is one that satisfies $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Then f has a fixed point.*

Proof. For positive integers, n, define $f_n(x) = (1 - \frac{1}{n}) f(x)$. Then each f_n is a contraction mapping of X. Since X is compact, it is complete. By the contraction mapping principle each f_n has a fixed point, $x_n \in X$ and since X is compact x_n has a subsequence converging to say x. Taking limits as $n \to \infty$ shows x is fixed by f. □

We now state the Brouwer fixed point theorem.