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Petr Hájek, Michal Johanis

SMOOTH ANALYSIS IN BANACH SPACES

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Smooth Analysis in Banach Spaces

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Smooth Analysis in Banach Spaces

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Introduction

The purpose of this book is to lay down the foundations for the abstract theory of C^k-smoothness in infinite-dimensional real Banach spaces, and investigate its intimate connections with the structural properties of the underlying spaces.

The main objects of the theory are polynomials and C^k-smooth (including real analytic) mappings. In some sense, the most important result concerning C^k-smooth mappings is the Taylor formula, which takes the familiar form known from the theory of functions on $\mathbb{R}^n$. This formula unveils the prominent role played by polynomials in smoothness (especially higher smoothness) questions by way of approximating smooth functions in the neighbourhood of a point. In the infinite-dimensional setting this role is brought even further, as polynomials also provide the vital link with the structure of the underlying Banach space.

This explains why polynomials have received a great deal of attention in the present book. We have included plenty of results concerning polynomials with the intention to build up a supply of results and points of view that could be useful for the future development of the theory. The material is mostly organised according to the methods used. We study polynomials on $\mathbb{R}^n$ in connection with isometric theory of finite-dimensional spaces. It turns out that homogeneous polynomials are closely related to isometric subspaces of ℓ_p^n for p even. The theory of Chebyshev polynomials is used to obtain optimal estimates on the size of higher derivatives for a given polynomial. The theory of tensor products and (p, q)-summing operators is applied to obtain sharp estimates on the values of polynomial coefficients for polynomials from c_0 to ℓ_p. The concept of finite representability combined with powerful finite-dimensional results concerning type and cotype of Banach spaces, spreading models, and ultrapowers lead to strong structural results for Banach spaces admitting separating polynomials. All the above results and much more are covered in the first four chapters. The breadth of the above mentioned material precludes making our presentation completely self contained.

We decided, for the benefit of the reader, to include most of the needed auxiliary results (without proof) in the form of short survey paragraphs, or sometimes whole sections, which form an integral part of the text. This makes it possible for the reader to follow the text, without having to jump into an appendix or a specialised monograph, and keep track of the flow of ideas.

The remaining three chapters of the book are devoted to a detailed study of smooth mappings between Banach spaces. The properties being studied can be roughly divided into three main areas whose polynomial (and usually finite-dimensional) counterparts are covered in the first half of the book.

The first aspect are the structural properties implied by the existence of certain polynomials, resp. separating C^k-smooth functions (in the finite-dimensional setting this corresponds to the study of subspaces of ℓ_p^n). The second aspect is the supply of such mappings (this corresponds essentially to quantitative estimates on the coefficients), and finally the last aspect concerns approximation questions (this relates to the theory of algebras of sub-symmetric polynomials).

The Banach space c_0 plays a major role throughout the subject. From the technical point of view, this is due to the fact that its supply of polynomials (and uniformly smooth functions) is very small (they are weakly uniformly continuous), but on the other hand its supply of C^∞-smooth functions is very large (thanks to the phenomenon of functions that depend locally on finitely many coordinates). Moreover, it is a universal space for C^k-smooth embeddings. There are two important classes of spaces which share some of its important features. It is the class of polyhedral spaces, which are C^∞-smooth, c_0 saturated, and behave well with respect to taking subspaces. From the other side it is the class $\mathcal{W}$, which contains all $\mathcal{L}_\infty$-spaces and behaves rather well with respect to uniformly smooth mappings and quotients. The intersection of these two classes contains all isometric preduals of ℓ_1, in particular all $C(K)$ spaces, K countable compact. The examples of $\mathcal{L}_\infty$-spaces of Jean Bourgain and Freddy Delbaen do not contain c_0, so they are not polyhedral. On the other hand, Schreier's space is a polyhedral space (isomorphic to a subspaces of $C([0,\omega^\omega]))$ which admits a non-compact linear operator into ℓ_2, and hence it does not belong to class $\mathcal{W}$. In fact, Ioannis Gasparis constructed polyhedral spaces with quotient ℓ_2.

The existence of a separating C^k-smooth function on a Banach space has strong structural consequences. For example a Banach space admitting a C^2-smooth bump either contains c_0 or it is super-reflexive. So the study of higher smoothness naturally splits into two rather distinct extreme situations, which also rely on distinct techniques. This feature is repeated with respect to smooth mappings, whose supply again depends heavily on the geometry of the underlying space. The former case corresponds to $\mathcal{W}$-spaces, which admit only few uniformly smooth mappings. More precisely, if Y^{**} has no subspace isomorphic to c_0, then any uniformly smooth mappings from a $\mathcal{W}$-space into Y is weakly compact. On the other hand, super-reflexive spaces admit surjective polynomials onto any separable Banach space.

In the questions on approximability of functions and mappings the space c_0 plays a fundamental role thanks to being a universal C^k-smooth embedding space. However, for real analytic approximations the available methods again seem to distinguish between spaces with a separating polynomial or merely a separating analytic function (the c_0-type). At present it is not clear if this distinction can be overcome by improving the methods of proof.

Our book is focusing on the case of real Banach spaces, using the complex case only as a tool for dealing with analytic functions. We feel that the theory has now reached a certain level of maturity, in the sense that extreme cases of higher smoothness, i.e. the

c_0-like, or finite rank phenomena, and on the other hand the separating polynomial-like phenomena have been well-understood. Our rendering of the state of the theory is rather complete and up to date, and it appears for the first time in a book form. A large part is devoted to very recent developments, sometimes in an outline form. We believe that our book may serve as a rather complete reference book for the results and techniques in the area of smoothness in separable real Banach spaces. It can also be used as a textbook by advanced graduate students and active researchers with solid background in Banach space theory, and for this purpose we pose a number of open problems for independent research. The future now lies in investigation of general polynomials, and higher order approximations. This will most probably require a new set of techniques.

Let us proceed with a more detailed description of some highlighted results in the respective chapters. The first chapter contains an abstract theory of C^k-smoothness in any Banach space. We introduce multilinear mappings and polynomials, as higher derivatives are defined to be these objects. The main result in this respect is the Taylor formula which provides a fundamental link between smoothness, polynomials, and the underlying Banach space structure. We also need to introduce the complexification of spaces and polynomials. This is indispensable for the development of the theory of analytic functions, but also very convenient in other situations thanks to efficient averaging methods for complex polynomials. Our treatment of (real) analytic functions was inspired by papers of Jacek Bochnak and Józef Siciak. Analytic functions are defined as functions locally admitting power series expansion. There are several important characterisations, in particular by using finite-dimensional restrictions, and by using the complexified series. These characterisations are also an important tool in developing the theory. The theory depends heavily on results from one or several complex variables, which are mentioned without proof and applied through the notion of higher Gâteaux smoothness.

In brief summary, the first chapter gives a complete and self contained introduction to the subject, including the case of real analytic functions which to the best of our knowledge is for the first time in a book form. The remaining six chapters are best read consecutively. Chapters 2–4, which focus on properties of polynomials, cover all the necessary background needed for the remaining Chapters 5–7, which are dealing with general smooth mappings. Most of the material contained therein is also new to book form.

In Chapter 2 we focus on the duality theory for spaces of polynomials on finite-dimensional Banach spaces, in particular the cubature formulae representing positive functionals. We present a very simple and original proof of the well-known Chakalov theorem. We outline the proofs of cubature formulae related to Chebyshev polynomials on $\mathbb{R}$ and $\mathbb{R}^2$. This topic seems to be rather special but it leads to the proof of the Skalyga-Markov-type inequality, an optimal estimate on the size of higher derivatives of real polynomials. We continue by showing that every homogeneous polynomial is a finite sum of powers of functionals, which leads to the theory of isometric embeddings of finite-dimensional spaces into ℓ_p-spaces where p is even. We present, without proof, the recent result of Vladimir Leonidovich Dol'nikov and Roman Nikolaevich Karasev, and its close (but easier) relative, the Birch theorem. According to these results, a k-homogeneous polynomial on $\mathbb{R}^N$ can be restricted to a suitable $n(N,k)$-dimensional subspace E so that this restriction is equivalent to a power of the Euclidean norm on E. We cover the basic theory of identities for polynomials, which implies that a continuous function is a polynomial of degree at most n provided that its restriction to every affine one-dimensional subspace is also a polynomial of degree at most n. The last section is devoted to a relatively simple but powerful averaging method for treating polynomials and multilinear mappings, which leads to important estimates of polynomial coefficients. More sophisticated methods which combine probabilistic averaging with other techniques are treated in the subsequent chapter.

Chapter 3 is devoted to the study of polynomials between Banach spaces and the way they act on sequences from the initial space. This is a classical theme in the theory of linear operators, represented by the Dunford-Pettis property and the theory of p-summing operators. We start the chapter by outlining the basics on tensor products as well as symmetric tensor products and their duality with polynomial spaces. We then outline the concepts of uniform spaces and uniform continuity which play a key role throughout the whole book. We proceed by developing in detail the basic theory of weakly, weakly sequentially, and weakly uniformly continuous mappings and in particular polynomials. This investigation was initiated in the early papers of Aleksander Pełczyński and more systematically developed by Richard Martin Aron and his coauthors. An important contribution of Raymond A. Ryan was the introduction of symmetric tensor products into the subject, as well as the proof that every weakly compact polynomial from a space with the Dunford-Pettis property maps weakly Cauchy sequences to norm convergent ones.

In Banach spaces not containing ℓ_1 the weak sequential continuity on bounded sets coincides with full (or even uniform) continuity. This important fact results from Rosenthal's ℓ_1 theorem and other related results of Edward Wilfred Odell and results of Jean Bourgain, David H. Fremlin, and Michel Talagrand.

We introduce the language of the theory of (p,q)-summing operators, mention some connections with tensor products, and formulate some of the fundamental results of the theory, introducing also the notions of type and cotype. Using the theory of multiple $(p;1)$-summing operators, we give optimal estimates on the coefficients of polynomials in $\mathcal{P}({}^n c_0; \ell_p)$, following the recent work of Andreas Defant and Pablo Sevilla-Peris.

In Chapter 4 we apply the concept of finite representability to the study of asymptotic behaviour of polynomials and the linear structure of the underlying Banach spaces. We describe the ultrapower construction for a Banach space X, which leads to a much larger Banach space $(X)_{\mathcal{U}}$ that is finitely representable in X and that is well-suited for constructions of uniformly continuous mappings. As an application we show that uniformly smooth mappings can be extended into the bidual. By using the spreading models of X, which capture the asymptotic behaviour of infinite sequences in X, we study the upper and lower estimates of sequences, the Banach-Saks and the weak p-Banach-Saks properties. We proceed by showing that if X has a sub-symmetric basis and a separating d-homogeneous polynomial, then it is isomorphic to some ℓ_p for p even, and $d = kp$, $k \in \mathbb{N}$. As a corollary we obtain a fundamental result of Robert Deville stating that every Banach space with a separating polynomial contains a subspace isomorphic to ℓ_p, p even. We study in detail the restrictions of polynomials on ℓ_p to subspaces generated by suitable subsequences of the canonical basis. The main result in this direction, which goes well beyond the asymptotic but finite-dimensional results based on spreading models, claims that these restrictions are almost sub-symmetric. As a corollary to all these results we show that for an arbitrary polynomial P on X there is an infinite-dimensional subspace Y of X such that the restriction of P to Y is either separating or asymptotically zero in a strong sense.

The last sections are devoted to the study of algebras $\mathcal{A}_n(X)$ of polynomials generated by polynomials of degree at most n on a Banach space X. The main technical tool is a finite-dimensional lemma which claims that the symmetric polynomial $s_n^N(x) = \sum_{j=1}^N x_j^n$ on $\mathbb{R}^N$ is not in the uniform closure of a suitably defined sub-symmetric subalgebra of $\mathcal{A}_{n-1}(\mathbb{R}^N)$, provided that N is large enough. We proceed by applying this result to infinite-dimensional spaces using spreading model techniques. The main result implies in particular that $\overline{\mathcal{A}_1(\ell_p)} = \cdots = \overline{\mathcal{A}_{n-1}(\ell_p)} \subsetneq \overline{\mathcal{A}_n(\ell_p)} \subsetneq \overline{\mathcal{A}_{n+1}(\ell_p)} \subsetneq \cdots$, where $n = \lceil p \rceil$.

Chapter 5 is devoted to the detailed study of Banach spaces admitting smooth separating functions. An important set of tools for obtaining structural results are the variational principles. When applied to a given lower continuous and bounded below function f, they guarantee the existence of a point $x \in X$ and a smooth function g such that $f - g$ attains its minimum, sometimes in a strong sense. If f itself is a smooth function, then depending on the concrete conditions we may use the Taylor formula at x in order to obtain uniformly smooth separating functions on X, or at least some structural information about X. We describe two examples of this notion. Stegall's variational principle holds in every Banach space with the RNP and the function g can be chosen to be a functional from X^*. As a result, if a Banach space with the RNP admits twice Gâteaux smooth bump function, we conclude that X is super-reflexive and admits a norm with a power type 2 estimate on the modulus of smoothness.

The compact variational principle of Robert Deville and Marián Fabian, which was motivated by a paper of Jaroslav Pechanec, John H. M. Whitfield, and Václav Zizler, is a key tool for studying higher smoothness in Banach spaces. Its formulation is somewhat different from the general description given above, but it leads to similar applications. This principle (we prefer to avoid the precise formulation at this point) applies to Banach spaces which do not contain c_0, and it has lead to several strong structural results. The most important structural result is that if X has no subspace isomorphic to c_0 and has a C^k-smooth bump, for large enough k, then X is super-reflexive, admits a separating polynomial, and contains a subspace isomorphic to ℓ_p, p even.

Jaroslav Kurzweil has pioneered the field of higher smoothness by finding that the best order of smoothness of L_p-spaces is $C^{[p]}$, except in the case when p is even and the space admits a separating polynomial. In the latter case he constructed real analytic approximations for all continuous mappings from these spaces. In the rest of the chapter we compute the best order of smoothness for several classes of Banach spaces, notably the Orlicz spaces and the class of polyhedral Banach spaces.

In Chapter 6 we develop the theory of uniformly smooth mappings between Banach spaces. We study the relationship between uniform continuity and weak uniform continuity of the mapping and its higher derivatives. In particular, if a weakly uniformly continuous mapping has a uniformly continuous kth derivative, then this derivative is even weakly uniformly continuous. In Section 6.2 we introduce and study the important concept of bidual extension for $C^{k,+}$-smooth mappings from the unit ball of a Banach space X into Y to a $C^{k,+}$-smooth mapping from the unit ball of X^{**} into Y^{**}. This notion contains the classical bi-adjoint of a linear operator as a special case, but it is not completely canonical and in general depends on some parameters in the construction. This notion plays an important role later on in pushing some results for $C(K)$ spaces where K is scattered into the case of a general compact space K.

In Sections 6.3–6.6 we build the theory of $\mathcal{W}_\lambda$-spaces, i.e. all those X such that uniformly smooth functions in B_X map weakly Cauchy sequences in λB_X, for some $\lambda \in (0, 1]$, to convergent sequences. The main goal of this theory is to generalise some classical properties of linear operators from $C(K)$ spaces, described in Theorem 3.47, into the setting of uniformly smooth mappings. This objective is achieved in the main Theorem 6.57, which claims that weakly compact and uniformly smooth mappings from the unit ball of $C(K)$ spaces take weakly Cauchy sequences into norm convergent ones. The first step of the proof is to show the rigidity of ℓ_∞^n with respect to uniformly smooth functions. The second step consists of showing that $C(K)$, for K scattered, are $\mathcal{W}_1$-spaces. Next, we prove that the bidual extension of any uniformly smooth non-compact mapping in $\mathcal{C}^{1,+}(B_{c_0}; Y)$ has a point where the derivative is non-compact, and hence fixes a copy of c_0. This implies that Y^{**} has a subspace isomorphic to c_0. Moreover, non-compact uniformly smooth mappings from $C(K)$, K scattered, can always be reduced to a suitable subspace isomorphic to c_0 where the restriction remains non-compact. The case of a general $C(K)$ space requires another ingredient, Theorem 6.56, which claims that after passing to the bidual, weakly Cauchy sequences in $C(K)$ spaces are uniformly close to weakly Cauchy sequences in some $C([0, \alpha])$.

In Section 6.7 we give some rather general results on the ranges of smooth mappings (and derivatives of smooth functions) which illustrate that the theory of $\mathcal{W}$-spaces works under nearly optimal assumptions. Indeed, according to Theorem 6.69 no structural property of the initial space is generally preserved by surjective C^∞-smooth mappings from Banach spaces with property $\mathcal{B}$. In the rest of the chapter our attention shifts to smooth separating mappings from ℓ_p-spaces, proving strong structural result in this case, based on the notions of a separating mapping and harmonic behaviour.

Smooth approximations in Banach spaces are studied in the last chapter of our book. Unlike the finite-dimensional case, the unit ball of an infinite-dimensional Banach space X is not a compact set and hence it is easy to find uniformly continuous functions that are not uniformly approximable by polynomials on B_X. Interestingly enough, in the special case of $\mathcal{W}$-spaces X not containing ℓ_1, uniformly C^k-smooth functions can be uniformly approximated, together with their higher derivatives, by polynomials on B_X. This result underlines the extremely poor supply of uniformly smooth functions in these spaces, rather than the abundance of polynomials. Indeed, in this case all polynomials are weakly uniformly continuous on B_X. This is again a situation when smoothness properties take on very different shapes for c_0-like spaces (although X considered here need not even contain c_0!), and for super-reflexive spaces.

The main tool for studying C^k-smooth approximations of continuous mappings are C^k-smooth partitions of unity, or alternatively C^k-smooth embeddings into c_0. The problem becomes more challenging if additional conditions are put on the approximants, and the usual partitions of unity cannot be employed as they destroy the character of the approximating functions. This concerns for example the problem of smooth approximations preserving the Lipschitz constants, which is the first step for obtaining approximations together with higher derivatives, a problem that remains widely open. We finish the chapter by proving that if a separable Banach space admits a C^k-smooth equivalent norm, then every norm can be approximated on bounded sets by C^k-smooth renormings.

Let us make some remarks concerning the existing literature related to the subject of our book. Introduction to abstract smooth analysis (real or complex) forms part of Jean Dieudonné's book [Dieu], which was a great source of inspiration for us. The complex case, where all notions of smoothness coincide, has received more attention in monographs, e.g. in [Hill], [Din], [Mu], [Na]. We have relied much on Seán Dineen's book for its insights, historical comments, and ample references. The distinguishing feature of the real setting, as opposed to complex one, is the intricate role played by the Banach space structure and geometry for the supply of polynomials and C^k-smooth functions.

Concerning polynomials on finite dimension spaces, there are of course many sources, see e.g. [MMR], [BE]. Apparently the main interest in polynomials in the infinite-dimensional setting came from the study of analytic functions, see [Din] for historical comments.

Finally, it was the monograph of Robert Deville, Gilles Godefroy, and Václav Zizler [DGZ], which has undertaken a systematic study of smoothness (including higher smoothness) in the context of Banach space structure. Their book treats both the separable and non-separable situation (in the latter they are still relatively up to date, so we decided for the most part to omit it). Our book overlaps only to a small extent with [DGZ], but the problems posed therein played a decisive role for the subsequent development of the subject.

We mention that the first order smoothness theory is well covered (apart from [DGZ]) also in [Fab3], and the more introductory [FHHMZ]. The differentiability of Lipschitz mappings is treated in [LPT]. We are not covering any aspects of this direction of research. In a broader sense, our subject forms part of geometric non-linear analysis, a fast growing subject whose foundations are laid out in [BenLi].

Concerning prerequisites, most of the background material on the linear structural theory of Banach spaces can be found in [FHHMZ]. Deeper results from local theory, which play a crucial role throughout the book, can be found in [DJT]. These two references provide a solid background for the material presented in our book, but our theory draws on results from other areas as well. For the theory of Chebyshev polynomials we suggest [Ri], tensor products are treated in [DefFl] or [Ry3]. A detailed exposition of spreading models is in [BeaLa]. Structural theory of $C(K)$ and $\mathcal{L}_\infty$-spaces, which is used in Chapter 6, is covered in [LiTz1], and for $C(K)$ spaces also in [Ros3]. Structural theory of the classical Banach spaces is developed in more detail in [AK], [Dies2], [LiTz2], and [LiTz3]. For topological results we refer mostly to [Eng].

For an additional source of open problem we refer to [FMZ].

Acknowledgement

In the early 90's the first named author started his PhD research under the supervision of Václav Zizler, then at the University of Alberta in Edmonton. At that time Václav was finishing his book [DGZ], which has played a decisive role for the future development of this area, through its thorough exposition of the known results and techniques, but also through many attractive but approachable problems. Not only that. For the benefit of his students, Václav has compiled a list of additional 50 (later expanded to about 90) problems in the areas close to [DGZ], many of them related to higher smoothness. It was precisely this abundance of nice open problems and Václav's keen interest in any progress, that attracted the first named author to this field of research.

Some of the problems have been solved, and their solutions have lead to the theory presented in this book. Many remain still open, and have been again posed in the present work in the hope that they will do a similar service to the next generation of students.

We would like to thank Václav for all those years of support, interest, and friendship. We would also like to thank our closest colleagues and friends (in alphabetical order) Richard M. Aron, Robert Deville, Marián Fabian, Gilles Godefroy, Gilles Lancien, Vicente Montesinos, and Stanimir L. Troyanski for their important contribution to the present work in the form of open problems, shared knowledge, moral support, and opportunity for discussions and seminar presentations. Finally, we would like to thank our colleagues Michal Kraus and Luděk Zajíček for reading parts of the manuscript and supplying us with errata and critical comments.

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Notation

We fix some notation for objects and notions that the reader should be familiar with. By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ we denote the sets of natural numbers, integers, rational numbers, reals, and complex numbers respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathbb{R}^+$ we denote the set of positive real numbers and by $\mathbb{R}_0^+$ the set of non-negative real numbers. By $\overline{\mathbb{R}}$ we denote the extended real line, i.e. $\overline{\mathbb{R}} = [-\infty, +\infty]$. By $\mathbb{K}$ we denote the scalar field $\mathbb{R}$ or $\mathbb{C}$. We use the convention that a sum over an empty set is zero and a product over an empty set is equal to 1. Further, $x^0 = 1$ for any $x \in \mathbb{K}$. For $x \in \mathbb{R}$ we denote by $[x]$ the integer part of x, i.e. the unique number $k \in \mathbb{Z}$ satisfying $k \leq x < k+1$, by $\lceil x \rceil$ we denote the ceiling of x, i.e. the unique number $k \in \mathbb{Z}$ satisfying $k - 1 < x \leq k$.

For a set A we denote its cardinality by $|A|$ or card A. The cardinality of the continuum is denoted by $\mathfrak{c}$. By abusing the notation we write $\{x_\gamma\}_{\gamma \in \Gamma} \subset X$ meaning that $\{x_\gamma\}_{\gamma \in \Gamma}$ is a collection such that $x_\gamma \in X$ for each $\gamma \in \Gamma$.

Let (P, ρ) be a metric space. We denote $B(x, r) = \{y \in P;\ \rho(y, x) \leq r\}$ and $U(x, r) = \{y \in P;\ \rho(y, x) < r\}$ the closed, resp. open ball in P centred at $x \in P$ with radius $r > 0$. In case that it is necessary to distinguish the spaces in which the balls are taken, we will write $B_P(x, r)$, resp. $U_P(x, r)$. By B_X and U_X we denote the closed, resp. open unit ball of a normed linear space X. By S_X we denote the unit sphere of a normed linear space X. An interior of a set A in a topological space is denoted by $\operatorname{Int} A$, its boundary is denoted by ∂A.

Throughout the book we use the following convention: In each statement involving multiple vector spaces we assume that all the spaces are over the same field $\mathbb{K}$ if not specified otherwise. Furthermore, if not specified explicitly or in the beginning of the chapter or section, then the statement holds both for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. When we speak of a subspace of a Banach space, we always mean a closed subspace. General subspaces will be referred to as "linear subspaces". We define $\operatorname{span} \emptyset = \{0\}$. If X is a normed linear space with a Schauder basis $\{e_n\}$ and $x = \sum_{n=1}^{\infty} x_n e_n \in X$, then $\operatorname{supp} x = \{n \in \mathbb{N};\ x_n \neq 0\}$ is called a support of x; a finitely supported vector is a vector with finite support. An algebraic dual of a vector space is denoted by $X^\#$, a topological dual of a topological vector space by X^*. Inner product is denoted by $\langle x, y \rangle$, and similarly we denote the evaluation in duality by $\langle f, x \rangle$. Let X, Y be normed linear spaces. For simplicity we say that X contains Y if X has a subspace isomorphic to Y.

By $C(X; Y)$ we denote the set of continuous mappings between topological spaces X, Y. If Y is a topological vector space, then $C(X; Y)$ is a vector space. For functions, i.e. mappings into the scalars, we use a shortened notation $C(X) = C(X; \mathbb{K})$; from the context it should always be clear whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For a mapping $f\colon X \to Y$, where Y is a vector space, we denote $\operatorname{supp_o} f = f^{-1}(Y \setminus \{0\})$. If X is a topological space, then we denote $\operatorname{supp} f = \overline{\operatorname{supp_o} f}$. An L-Lipschitz mapping is a mapping that is Lipschitz with a constant L. By χ_A we denote the characteristic function of the set A.

If we say measure or Borel measure, we always mean a non-negative measure. On the other hand, Radon measure means scalar-valued Radon measure. By $\operatorname{supp} \mu$ we denote the support of a Borel measure μ. The n-dimensional Lebesgue measure will be denoted by λ_n, or just λ if the dimension is clear from the context.

All topological spaces in this volume are automatically and without mention assumed to be Hausdorff.

Contents

Introduction vii

Chapter 1. Fundamental properties of smoothness 1
1. Multilinear mappings and polynomials 2
2. Complexification 22
3. Fréchet smoothness 26
4. Taylor polynomial 40
5. Smoothness classes 54
6. Power series and their convergence 61
7. Complex mappings 68
8. Analytic mappings 69
9. Notes and remarks 79

Chapter 2. Basic properties of polynomials on $\mathbb{R}^n$ 82
1. Spaces of polynomials on $\mathbb{R}^n$ 83
2. Cubature formulae 87
3. Estimates related to Chebyshev polynomials 96
4. Polynomials and L_p-norms on $\mathbb{R}^n$ 101
5. Polynomial identities 111
6. Estimates of coefficients of polynomials 119
7. Notes and remarks 126

Chapter 3. Weak continuity of polynomials and estimates of coefficients 130
1. Tensor products and spaces of multilinear mappings 131
2. Weak continuity and spaces of polynomials 135
3. Weak continuity and ℓ_1 159
4. (p,q)-summing operators 165
5. Estimates of coefficients of multilinear mappings 170
6. Bohr radius 176
7. Notes and remarks 179

Chapter 4. Asymptotic properties of polynomials 184
1. Finite representability and ultraproducts 185
2. Spreading models 192
3. Polynomials and p-estimates 198
4. Separating polynomials. Symmetric and sub-symmetric polynomials 203

5. Stabilisation of polynomials 214
6. Sub-symmetric polynomials on $\mathbb{R}^n$ 223
7. Polynomial algebras on Banach spaces 229
8. Notes and remarks 235

Chapter 5. Smoothness and structure 239
1. Convex functions 240
2. Smooth bumps and structure I 257
3. Smooth variational principles 263
4. Smooth bumps and structure II 270
5. Local dependence on finitely many coordinates 278
6. Isomorphically polyhedral spaces 290
7. L_p spaces 294
8. $C(K)$ spaces 304
9. Orlicz spaces 307
10. Notes and remarks 321

Chapter 6. Structural behaviour of smooth mappings 328
1. Weak uniform continuity and higher smoothness 329
2. Bidual extensions 333
3. Class $\mathcal{W}$ 338
4. Uniformly smooth mappings from $C(K)$, K scattered 349
5. Uniformly smooth mappings from $\mathcal{W}$-spaces 357
6. Fixing the canonical basis of c_0 362
7. Ranges of smooth mappings 365
8. Harmonic behaviour of smooth mappings 369
9. Notes and remarks 378

Chapter 7. Smooth approximation 382
1. Separation 383
2. Approximation by polynomials 391
3. Approximation by real-analytic mappings 399
4. Infimal convolution 407
5. Approximation of continuous mappings and partitions of unity 411
6. Non-linear embeddings into $c_0(\Gamma)$ 422
7. Approximation of Lipschitz mappings 428
8. Approximation of C^1-smooth mappings 452
9. Approximation of norms 456
10. Notes and remarks 462

Bibliography 467

Notation 490

Index 493