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Alan F. Beardon

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Alan F. Beardon

The Geometry of Discrete Groups

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(continued after index)

To Toni

Preface

This text is intended to serve as an introduction to the geometry of the action of discrete groups of Möbius transformations. The subject matter has now been studied with changing points of emphasis for over a hundred years, the most recent developments being connected with the theory of 3-manifolds: see, for example, the papers of Poincaré [77] and Thurston [101]. About 1940, the now well-known (but virtually unobtainable) Fenchel–Nielsen manuscript appeared. Sadly, the manuscript never appeared in print, and this more modest text attempts to display at least some of the beautiful geometrical ideas to be found in that manuscript, as well as some more recent material.

The text has been written with the conviction that geometrical explanations are essential for a full understanding of the material and that however simple a matrix proof might seem, a geometric proof is almost certainly more profitable. Further, wherever possible, results should be stated in a form that is invariant under conjugation, thus making the intrinsic nature of the result more apparent. Despite the fact that the subject matter is concerned with groups of isometries of hyperbolic geometry, many publications rely on Euclidean estimates and geometry. However, the recent developments have again emphasized the need for hyperbolic geometry, and I have included a comprehensive chapter on analytical (not axiomatic) hyperbolic geometry. It is hoped that this chapter will serve as a "dictionary" of formulae in plane hyperbolic geometry and as such will be of interest and use in its own right. Because of this, the format is different from the other chapters: here, there is a larger number of shorter sections, each devoted to a particular result or theme.

The text is intended to be of an introductory nature, and I make no apologies for giving detailed (and sometimes elementary) proofs. Indeed, many geometric errors occur in the literature and this is perhaps due, to some extent, to an omission of the details. I have kept the prerequisites to a minimum and, where it seems worthwhile, I have considered the same topic from different points of view. In part, this is in recognition of the fact that readers do not always read the pages sequentially. The list of references is not comprehensive and I have not always given the original source of a result. For ease of reference, Theorems, Definitions, etc., are numbered collectively in each section (2.4.1, 2.4.2, . . .).

I owe much to many colleagues and friends with whom I have discussed the subject matter over the years. Special mention should be made, however, of P. J. Nicholls and P. Waterman who read an earlier version of the manuscript, Professor F. W. Gehring who encouraged me to write the text and conducted a series of seminars on parts of the manuscript, and the notes and lectures of L. V. Ahlfors. The errors that remain are mine.

Cambridge, 1982 ALAN F. BEARDON

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CHAPTER 1

Preliminary Material

§1.1. Notation

We use the following notation. First, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the integers, the rationals, the real and complex numbers respectively: $\mathbb{H}$ denotes the set of quaternions (Section 2.4).

As usual, $\mathbb{R}^n$ denotes Euclidean n-space, a typical point in this being $x = (x_1, \ldots, x_n)$ with

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Note that if $y > 0$, then $y^{1/2}$ denotes the positive square root of y. The standard basis of $\mathbb{R}^n$ is $e_1, \ldots, e_n$ where, for example, $e_1 = (1, 0, \ldots, 0)$. Certain subsets of $\mathbb{R}^n$ warrant special mention, namely

$$B^n = \{x \in \mathbb{R}^n : |x| < 1\},$$

$$H^n = \{x \in \mathbb{R}^n : x_n > 0\},$$

and

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

In the case of $\mathbb{C}$ (identified with $\mathbb{R}^2$) we shall use Δ and $\partial\Delta$ for the unit disc and unit circle respectively.

The notation $x \mapsto x^2$ (for example) denotes the function mapping x to x^2: the domain will be clear from the context. Functions (maps or transformations) act on the *left*: for brevity, the image $f(x)$ is often written as fx (omitting brackets). The composition of functions is written as fg: this is the map $x \mapsto f(g(x))$.