Graduate Texts in Mathematics

Tom M. Apostol

Modular Functions and Dirichlet Series in Number Theory

Second Edition

数论中的模函数和狄利克莱级数

第2版

Springer

世界图书出版公司
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Second Edition

With 25 Illustrations

Springer

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Mathematics Subject Classification (2000): 11-01, 11FXX

Library of Congress Cataloging-in-Publication Data
Apostol, Tom M.
Modular functions and Dirichlet series in number theory/Tom M. Apostol.—2nd ed.
p. cm.—(Graduate texts in mathematics; 41)
Includes bibliographical references.
ISBN 0-387-97127-0 (alk. paper)
1. Number theory. 2. Functions, Elliptic. 3. Functions, Modular. 4. Series, Dirichlet. I. Title. II. Series.
QA241.A62 1990
512'.7—dc20 89-21760

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ISBN 0-387-97127-0
ISBN 3-540-97127-0 SPIN 10841555

Springer-Verlag New York Berlin Heidelberg
A member of BertelsmannSpringer Science+Business Media GmbH

图书在版编目（CIP）数据

数论中的模函数和狄利克莱级数＝Modular Functions and Dirichlet Series in Number Theory：第2版：英文/（美）阿波斯托尔（Apostol，T. M.）著．—北京：世界图书出版公司北京公司，2009. 4

ISBN 978-7-5100-0440-7

Ⅰ．数… Ⅱ．阿… Ⅲ．①模函数—高等学校—教材—英文②狄利克雷级数—高等学校—教材—英文 Ⅳ．0174．1 0156．4

中国版本图书馆 CIP 数据核字（2009）第 048129 号

书　　号：	Modular Functions and Dirichlet Series in Number Theory 2nd ed.
作　　者：	Tom M. Apostol
中 译 名：	数论中的模函数和狄利克莱级数 第2版
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街137号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24开
印　　张：	9.[illegible]
版　　次：	2009年04月
版权登记：	图字：01-2009-0689
书　　号：	978-7-5100-0440-7/O·655　　定　　价：35.00元

世界图书出版公司北京公司已获得 Springer 授权在中国大陆独家重印发行

Preface

This is the second volume of a 2-volume textbook* which evolved from a course (Mathematics 160) offered at the California Institute of Technology during the last 25 years.

The second volume presupposes a background in number theory comparable to that provided in the first volume, together with a knowledge of the basic concepts of complex analysis.

Most of the present volume is devoted to elliptic functions and modular functions with some of their number-theoretic applications. Among the major topics treated are Rademacher's convergent series for the partition function, Lehner's congruences for the Fourier coefficients of the modular function $j(\tau)$, and Hecke's theory of entire forms with multiplicative Fourier coefficients. The last chapter gives an account of Bohr's theory of equivalence of general Dirichlet series.

Both volumes of this work emphasize classical aspects of a subject which in recent years has undergone a great deal of modern development. It is hoped that these volumes will help the nonspecialist become acquainted with an important and fascinating part of mathematics and, at the same time, will provide some of the background that belongs to the repertory of every specialist in the field.

This volume, like the first, is dedicated to the students who have taken this course and have gone on to make notable contributions to number theory and other parts of mathematics.

T. M. A.
January, 1976

* The first volume is in the Springer-Verlag series Undergraduate Texts in Mathematics under the title *Introduction to Analytic Number Theory*.

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Preface to the Second Edition

The major change is an alternate treatment of the transformation formula for the Dedekind eta function, which appears in a five-page supplement to Chapter 3, inserted at the end of the book (just before the Bibliography). Otherwise, the second edition is almost identical to the first. Misprints have been repaired, there are minor changes in the Exercises, and the Bibliography has been updated.

T. M. A.
July, 1989

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Elliptic functions 1

1.1 Introduction

Additive number theory is concerned with expressing an integer n as a sum of integers from some given set S. For example, S might consist of primes, squares, cubes, or other special numbers. We ask whether or not a given number can be expressed as a sum of elements of S and, if so, in how many ways this can be done.

Let $f(n)$ denote the number of ways n can be written as a sum of elements of S. We ask for various properties of $f(n)$, such as its asymptotic behavior for large n. In a later chapter we will determine the asymptotic value of the partition function $p(n)$ which counts the number of ways n can be written as a sum of positive integers $\leq n$.

The partition function $p(n)$ and other functions of additive number theory are intimately related to a class of functions in complex analysis called *elliptic modular functions*. They play a role in additive number theory analogous to that played by Dirichlet series in multiplicative number theory. The first three chapters of this volume provide an introduction to the theory of elliptic modular functions. Applications to the partition function are given in Chapter 5.

We begin with a study of doubly periodic functions.

1.2 Doubly periodic functions

A function f of a complex variable is called *periodic* with period ω if

$$f(z + \omega) = f(z)$$

whenever z and $z + \omega$ are in the domain of f. If ω is a period, so is $n\omega$ for every integer n. If ω_1 and ω_2 are periods, so is $m\omega_1 + n\omega_2$ for every choice of integers m and n.

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Definition. A function f is called *doubly periodic* if it has two periods ω_1 and ω_2 whose ratio ω_2/ω_1 is not real.

We require that the ratio be nonreal to avoid degenerate cases. For example, if ω_1 and ω_2 are periods whose ratio is real and rational it is easy to show that each of ω_1 and ω_2 is an integer multiple of the same period. In fact, if $\omega_2/\omega_1 = a/b$, where a and b are relatively prime integers, then there exist integers m and n such that $mb + na = 1$. Let $\omega = m\omega_1 + n\omega_2$. Then ω is a period and we have

$$\omega = \omega_1\left(m + n\frac{\omega_2}{\omega_1}\right) = \omega_1\left(m + n\frac{a}{b}\right) = \frac{\omega_1}{b}(mb + na) = \frac{\omega_1}{b},$$

so $\omega_1 = b\omega$ and $\omega_2 = a\omega$. Thus both ω_1 and ω_2 are integer multiples of ω.

If the ratio ω_2/ω_1 is real and irrational it can be shown that f has arbitrarily small periods (see Theorem 7.12). A function with arbitrarily small periods is constant on every open connected set on which it is analytic. In fact, at each point of analyticity of f we have

$$f'(z) = \lim_{z_n \to 0} \frac{f(z + z_n) - f(z)}{z_n},$$

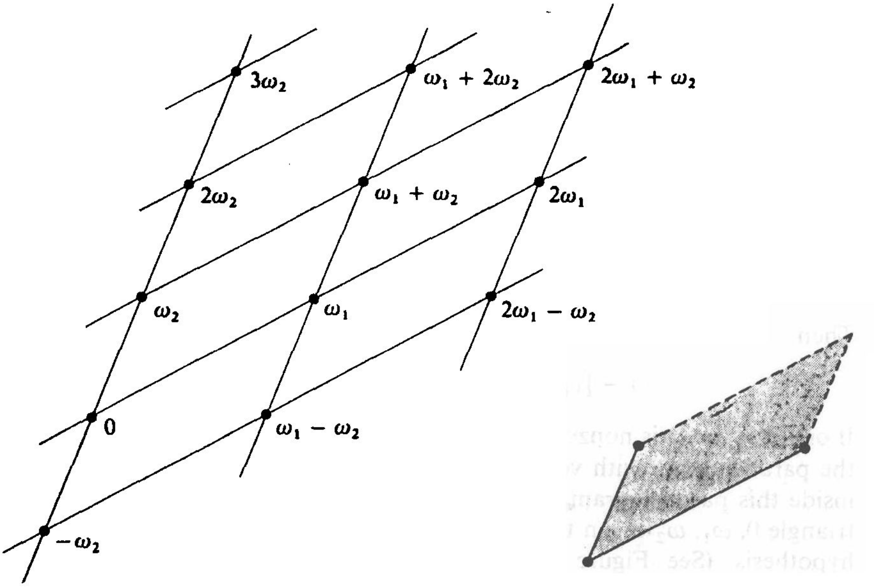
where $\{z_n\}$ is any sequence of nonzero complex numbers tending to 0. If f has arbitrarily small periods we can choose $\{z_n\}$ to be a sequence of periods tending to 0. Then $f(z + z_n) = f(z)$ and hence $f'(z) = 0$. In other words, $f'(z) = 0$ at each point of analyticity of f, hence f must be constant on every open connected set in which f is analytic.

1.3 Fundamental pairs of periods

Definition. Let f have periods ω_1, ω_2 whose ratio ω_2/ω_1 is not real. The pair (ω_1, ω_2) is called a *fundamental pair* if every period of f is of the form $m\omega_1 + n\omega_2$, where m and n are integers.

Every fundamental pair of periods ω_1, ω_2 determines a network of parallelograms which form a tiling of the plane. These are called *period parallelograms*. An example is shown in Figure 1.1a. The vertices are the periods $\omega = m\omega_1 + n\omega_2$. It is customary to consider two intersecting edges and their point of intersection as the only boundary points belonging to the period parallelogram, as shown in Figure 1.1b.

Notation. If ω_1 and ω_2 are two complex numbers whose ratio is not real we denote by $\Omega(\omega_1, \omega_2)$, or simply by Ω, the set of all linear combinations $m\omega_1 + n\omega_2$, where m and n are arbitrary integers. This is called the lattice generated by ω_1 and ω_2.



(a) (b)

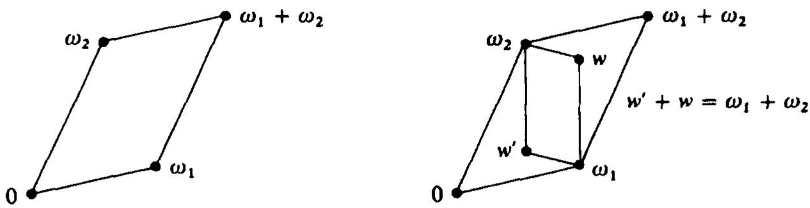
Figure 1.1

Theorem 1.1. *If (ω_1, ω_2) is a fundamental pair of periods, then the triangle with vertices 0, ω_1, ω_2 contains no further periods in its interior or on its boundary. Conversely, any pair of periods with this property is fundamental.*

PROOF. Consider the parallelogram with vertices 0, ω_1, $\omega_1 + \omega_2$, and ω_2, shown in Figure 1.2a. The points inside or on the boundary of this parallelogram have the form

$$z = \alpha\omega_1 + \beta\omega_2,$$

where $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Among these points the only periods are 0, ω_1, ω_2, and $\omega_1 + \omega_2$, so the triangle with vertices 0, ω_1, ω_2 contains no periods other than the vertices.



(a) (b)

Figure 1.2

Conversely, suppose the triangle 0, ω_1, ω_2 contains no periods other than the vertices, and let ω be any period. We are to show that $\omega = m\omega_1 + n\omega_2$ for some integers m and n. Since ω_2/ω_1 is nonreal the numbers ω_1 and ω_2 are linearly independent over the real numbers, hence

$$\omega = t_1\omega_1 + t_2\omega_2$$

where t_1 and t_2 are real. Now let $[t]$ denote the greatest integer $\leq t$ and write

$$t_1 = [t_1] + r_1, t_2 = [t_2] + r_2, \text{ where } 0 \leq r_1 < 1 \text{ and } 0 \leq r_2 < 1.$$

Then

$$\omega - [t_1]\omega_1 - [t_2]\omega_2 = r_1\omega_1 + r_2\omega_2.$$

If one of r_1 or r_2 is nonzero, then $r_1\omega_1 + r_2\omega_2$ will be a period lying inside the parallelogram with vertices 0, ω_1, ω_2, $\omega_1 + \omega_2$. But if a period w lies inside this parallelogram then either w or $\omega_1 + \omega_2 - w$ will lie inside the triangle 0, ω_1, ω_2 or on the diagonal joining ω_1 and ω_2, contradicting the hypothesis. (See Figure 1.2b.) Therefore $r_1 = r_2 = 0$ and the proof is complete. □

Definition. Two pairs of complex numbers (ω_1, ω_2) and (ω_1', ω_2'), each with nonreal ratio, are called *equivalent* if they generate the same lattice of periods; that is, if $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$.

The next theorem, whose proof is left as an exercise for the reader, describes a fundamental relation between equivalent pairs of periods.

Theorem 1.2. *Two pairs (ω_1, ω_2) and (ω_1', ω_2') are equivalent if, and only if, there is a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and determinant $ad - bc = \pm 1$, such that*

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix},$$

or, in other words,

$$\omega_2' = a\omega_2 + b\omega_1,$$
$$\omega_1' = c\omega_2 + d\omega_1.$$

1.4 Elliptic functions

Definition. A function f is called *elliptic* if it has the following two properties:

(a) f is doubly periodic.

(b) f is meromorphic (its only singularities in the finite plane are poles).

Constant functions are trivial examples of elliptic functions. Later we shall give examples of nonconstant elliptic functions, but first we derive some fundamental properties common to all elliptic functions.

Theorem 1.3. *A nonconstant elliptic function has a fundamental pair of periods.*

PROOF. If f is elliptic the set of points where f is analytic is an open connected set. Also, f has two periods with nonreal ratio. Among all the nonzero periods of f there is at least one whose distance from the origin is minimal (otherwise f would have arbitrarily small nonzero periods and hence would be constant). Let ω be one of the nonzero periods nearest the origin. Among all the periods with modulus $|\omega|$ choose the one with smallest nonnegative argument and call it ω_1. (Again, such a period must exist otherwise there would be arbitrarily small nonzero periods.) If there are other periods with modulus $|\omega_1|$ besides ω_1 and $-\omega_1$, choose the one with smallest argument greater than that of ω_1 and call this ω_2. If not, find the next larger circle containing periods $\neq n\omega_1$ and choose that one of smallest nonnegative argument. Such a period exists since f has two noncollinear periods. Calling this one ω_2 we have, by construction, no periods in the triangle $0, \omega_1, \omega_2$ other than the vertices, hence the pair (ω_1, ω_2) is fundamental. □

If f and g are elliptic functions with periods ω_1 and ω_2 then their sum, difference, product and quotient are also elliptic with the same periods. So, too, is the derivative f'.

Because of periodicity, it suffices to study the behavior of an elliptic function in any period parallelogram.

Theorem 1.4. *If an elliptic function f has no poles in some period parallelogram, then f is constant.*

PROOF. If f has no poles in a period parallelogram, then f is continuous and hence bounded on the closure of the parallelogram. By periodicity, f is bounded in the whole plane. Hence, by Liouville's theorem, f is constant. □

Theorem 1.5. *If an elliptic function f has no zeros in some period parallelogram, then f is constant.*

PROOF. Apply Theorem 1.4 to the reciprocal $1/f$. □

Note. Sometimes it is inconvenient to have zeros or poles on the boundary of a period parallelogram. Since a meromorphic function has only a finite number of zeros or poles in any bounded portion of the plane, a period parallelogram can always be translated to a congruent parallelogram with no zeros or poles on its boundary. Such a translated parallelogram, with no zeros or poles on its boundary, will be called a *cell*. Its vertices need not be periods.

Theorem 1.6. *The contour integral of an elliptic function taken along the boundary of any cell is zero.*

PROOF. The integrals along parallel edges cancel because of periodicity. □

Theorem 1.7. *The sum of the residues of an elliptic function at its poles in any period parallelogram is zero.*

PROOF. Apply Cauchy's residue theorem to a cell and use Theorem 1.6. □

Note. Theorem 1.7 shows that an elliptic function which is not constant has at least two simple poles or at least one double pole in each period parallelogram.

Theorem 1.8. *The number of zeros of an elliptic function in any period parallelogram is equal to the number of poles, each counted with multiplicity.*

PROOF. The integral

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz,$$

taken around the boundary C of a cell, counts the difference between the number of zeros and the number of poles inside the cell. But f'/f is elliptic with the same periods as f, and Theorem 1.6 tells us that this integral is zero. □

Note. The number of zeros (or poles) of an elliptic function in any period parallelogram is called the *order* of the function. Every nonconstant elliptic function has order ≥ 2.

1.5 Construction of elliptic functions

We turn now to the problem of constructing a nonconstant elliptic function. We prescribe the periods and try to find the simplest elliptic function having these periods. Since the order of such a function is at least 2 we need a second order pole or two simple poles in each period parallelogram. The two possibilities lead to two theories of elliptic functions, one developed by Weierstrass, the other by Jacobi. We shall follow Weierstrass, whose point of departure is the construction of an elliptic function with a pole of order 2 at $z = 0$ and hence at every period. Near each period ω the principal part of the Laurent expansion must have the form

$$\frac{A}{(z-\omega)^2} + \frac{B}{z-\omega}.$$

For simplicity we take $A = 1, B = 0$. Since we want such an expansion near each period ω it is natural to consider a sum of terms of this type,

$$\sum_{\omega} \frac{1}{(z - \omega)^2}$$

summed over all the periods $\omega = m\omega_1 + n\omega_2$. For fixed $z \neq \omega$ this is a double series, summed over m and n. The next two lemmas deal with convergence properties of double series of this type. In these lemmas we denote by Ω the set of all linear combinations $m\omega_1 + n\omega_2$, where m and n are arbitrary integers.

Lemma 1. *If* α *is real the infinite series*

$$\sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \frac{1}{\omega^\alpha}$$

converges absolutely if, and only if, $\alpha > 2$.

Proof. Refer to Figure 1.3 and let r and R denote, respectively, the minimum and maximum distances from 0 to the parallelogram shown. If ω is any of the 8 nonzero periods shown in this diagram we have

$$r \leq |\omega| \leq R \qquad \text{(for 8 periods } \omega\text{).}$$

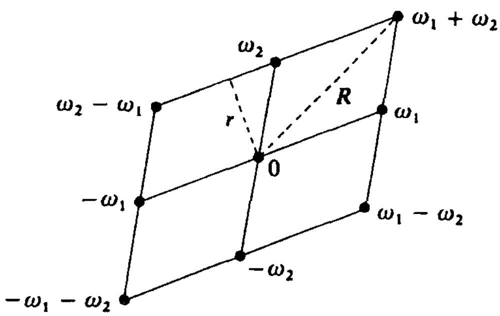


Figure 1.3

In the next concentric layer of periods surrounding these 8 we have $2 \cdot 8 = 16$ new periods satisfying the inequalities

$$2r \leq |\omega| \leq 2R \qquad \text{(for 16 new periods } \omega\text{).}$$

In the next layer we have $3 \cdot 8 = 24$ new periods satisfying

$$3r \leq |\omega| \leq 3R \qquad \text{(for 24 new periods } \omega\text{),}$$

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