

**Introduction
to
Perturbation
Techniques**

Introduction to Perturbation Techniques

ALI HASAN NAYFEH

*University Distinguished Professor
Virginia Polytechnic Institute and State University
Blacksburg, Virginia
and
Yarmouk University, Irbid, Jordan*

A WILEY-INTERSCIENCE PUBLICATION

JOHN WILEY & SONS

New York • Chichester • Brisbane • Toronto

Copyright © 1981 by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

Reproduction or translation of any part of this work beyond that permitted by Sections 107 or 108 of the 1976 United States Copyright Act without the permission of the copyright owner is unlawful. Requests for permission or further information should be addressed to the Permissions Department, John Wiley & Sons, Inc.

Library of Congress Cataloging in Publication Data:

Nayfeh, Ali Hasan, 1933-

Introduction to perturbation techniques.

"A Wiley-Interscience publication."

Bibliography: p.

Includes index.

1. Differential equations—Numerical solutions.
2. Equations—Numerical solutions. 3. Perturbation (Mathematics) I. Title.

QA371.N32 515.3'5 80-15233

ISBN 0-471-08033-0

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Preface

Many of the problems facing physicists, engineers, and applied mathematicians involve such difficulties as nonlinear governing equations, variable coefficients, and nonlinear boundary conditions at complex known or unknown boundaries that preclude solving them exactly. Consequently, solutions are approximated using numerical techniques, analytic techniques, and combinations of both. Foremost among the analytic techniques are the systematic methods of perturbations (asymptotic expansions) in terms of a small or a large parameter or coordinate. This book is concerned only with these perturbation techniques.

The author's book *Perturbation Methods* presents in a unified way an account of most of the perturbation techniques, pointing out their similarities, differences, and advantages, as well as their limitations. Although the techniques are described by means of examples that start with simple ordinary equations that can be solved exactly and progress toward complex partial-differential equations, the material is concise and advanced and therefore is intended for researchers and advanced graduate students only. The purpose of this book, however, is to present the material in an elementary way that makes it easily accessible to advanced undergraduates and first-year graduate students in a wide variety of scientific and engineering fields. As a result of teaching perturbation methods for eight years to first-year and advanced graduate students at Virginia Polytechnic Institute and State University, I have selected a limited number of techniques and amplified their description considerably. Also I have attempted to answer the questions most frequently raised by my students. The techniques are described by means of simple examples that consist mainly of algebraic and ordinary-differential equations.

The material in Chapters 3 and 15 and Appendices A and B cannot be found in *Perturbation Methods*. Chapter 3 discusses asymptotic expansions of integrals. Chapter 15 is devoted to the determination of the adjoints of homogeneous linear equations (algebraic, ordinary-differential, partial-differential, and integral equations) and the solvability conditions of linear inhomogeneous problems. Appendix A summarizes trigonometric identities, and Appendix B summarizes the properties of linear ordinary-differential equations and describes the symbolic method of determining the solutions of homogeneous and inhomogeneous ordinary-differential equations with constant coefficients.

The reader should have a background in calculus and elementary ordinary differential equations.

Each chapter contains a number of exercises. For more exercises, the reader is referred to *Perturbation Methods* by Nayfeh and *Nonlinear Oscillations* by Nayfeh and Mook. Since this book is elementary, only a list of the pertinent books is included in the bibliography without any attempt of citing them in the text.

I am indebted to K. R. Asfar and D. T. Mook for reading the whole manuscript and to L. Watson, M. Williams, C. Prather, S. A. Ragab, I. Wickman, A. Yen, Y. Liu, H. Reed, J. Dederer, Y. Ma, and W. S. Saric for reading parts of the manuscript. Many of the figures were drawn by T. H. Nayfeh, K. R. Asfar, I. Wickman, T. Duniyak, and T. McCawly; and I wish to express my appreciation to them. Finally, I wish to thank Patty Belcher, Janet Bryant, and Sharon Larkins for typing the manuscript.

ALI HASAN NAYFEH

Blacksburg, Virginia
April 1980

Contents

1	INTRODUCTION	1
1.1	Dimensional Analysis, 1	
1.2	Expansions, 10	
1.3	Gauge Functions, 12	
1.4	Order Symbols, 17	
1.5	Asymptotic Series, 18	
1.6	Asymptotic Expansions and Sequences, 22	
1.7	Convergent Versus Asymptotic Series, 23	
1.8	Elementary Operations on Asymptotic Expansions, 24	
	Exercises, 24	
2	ALGEBRAIC EQUATIONS	28
2.1	Quadratic Equations, 28	
2.2	Cubic Equations, 39	
2.3	Higher-Order Equations, 43	
2.4	Transcendental Equations, 45	
	Exercises, 48	
3	INTEGRALS	51
3.1	Expansion of Integrands, 52	
3.2	Integration by Parts, 56	
3.3	Laplace's Method, 65	
3.4	The Method of Stationary Phase, 79	

x CONTENTS

- 3.5 The Method of Steepest Descent, 88**
- Exercises, 101**

4 THE DUFFING EQUATION

107

- 4.1 The Straightforward Expansion, 109**
- 4.2 Exact Solution, 113**
- 4.3 The Lindstedt-Poincaré Technique, 118**
- 4.4 The Method of Renormalization, 121**
- 4.5 The Method of Multiple Scales, 122**
- 4.6 Variation of Parameters, 127**
- 4.7 The Method of Averaging, 129**
- Exercises, 131**

5 THE LINEAR DAMPED OSCILLATOR

134

- 5.1 The Straightforward Expansion, 135**
- 5.2 Exact Solution, 136**
- 5.3 The Lindstedt-Poincaré Technique, 139**
- 5.4 The Method of Multiple Scales, 142**
- 5.5 The Method of Averaging, 144**
- Exercises, 146**

6 SELF-EXCITED OSCILLATORS

147

- 6.1 The Straightforward Expansion, 148**
- 6.2 The Method of Renormalization, 151**
- 6.3 The Method of Multiple Scales, 152**
- 6.4 The Method of Averaging, 155**
- Exercises, 157**

7 SYSTEMS WITH QUADRATIC AND CUBIC NONLINEARITIES

159

- 7.1 The Straightforward Expansion, 160**

7.2	The Method of Renormalization, 162	
7.3	The Lindstedt-Poincaré Technique, 164	
7.4	The Method of Multiple Scales, 166	
7.5	The Method of Averaging, 168	
7.6	The Generalized Method of Averaging, 169	
7.7	The Krylov-Bogoliubov-Mitropolsky Technique, 173	
	Exercises, 175	
8	GENERAL WEAKLY NONLINEAR SYSTEMS	177
8.1	The Straightforward Expansion, 177	
8.2	The Method of Renormalization, 179	
8.3	The Method of Multiple Scales, 181	
8.4	The Method of Averaging, 182	
8.5	Applications, 184	
	Exercises, 188	
9	FORCED OSCILLATIONS OF THE DUFFING EQUATION	190
9.1	The Straightforward Expansion, 191	
9.2	The Method of Multiple Scales, 193	
9.2.1	Secondary Resonances, 193	
9.2.2	Primary Resonance, 205	
9.3	The Method of Averaging, 209	
9.3.1	Secondary Resonances, 209	
9.3.2	Primary Resonance, 212	
	Exercises, 213	
10	MULTIFREQUENCY EXCITATIONS	216
10.1	The Straightforward Expansion, 216	
10.2	The Method of Multiple Scales, 219	
10.2.1	The Case $\omega_2 + \omega_1 \approx 1$, 220	

xii CONTENTS

10.2.2	The Case $\omega_2 - \omega_1 \approx 1$ and $\omega_1 \approx 2$, 222	
10.3	The Method of Averaging, 226	
10.3.1	The Case $\omega_1 + \omega_2 \approx 1$, 230	
10.3.2	The Case $\omega_2 - \omega_1 \approx 1$ and $\omega_1 \approx 2$, 230	
	Exercises, 230	
11	THE MATHIEU EQUATION	234
11.1	The Straightforward Expansion, 235	
11.2	The Floquet Theory, 236	
11.3	The Method of Strained Parameters, 243	
11.4	Whittaker's Method, 247	
11.5	The Method of Multiple Scales, 249	
11.6	The Method of Averaging, 253	
	Exercises, 254	
12	BOUNDARY-LAYER PROBLEMS	257
12.1	A Simple Example, 257	
12.2	The Method of Multiple Scales, 268	
12.3	The Method of Matched Asymptotic Expansions, 270	
12.4	Higher Approximations, 279	
12.5	Equations with Variable Coefficients, 284	
12.6	Problems with Two Boundary Layers, 296	
12.7	Multiple Decks, 304	
12.8	Nonlinear Problems, 307	
	Exercises, 320	
13	LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS	325
13.1	First-Order Scalar Equations, 326	
13.2	Second-Order Equations, 329	
13.3	Solutions Near Regular Singular Points, 331	

13.4	Singularity at Infinity, 342	
13.5	Solutions Near an Irregular Singular Point, 344	
	Exercises, 355	
14	DIFFERENTIAL EQUATIONS WITH A LARGE PARAMETER	360
14.1	The WKB Approximation, 361	
14.2	The Liouville-Green Transformation, 364	
14.3	Eigenvalue Problems, 366	
14.4	Equations with Slowly Varying Coefficients, 369	
14.5	Turning-Point Problems, 370	
14.6	The Langer Transformation, 375	
14.7	Eigenvalue Problems with Turning Points, 379	
	Exercises, 383	
15	SOLVABILITY CONDITIONS	388
15.1	Algebraic Equations, 389	
15.2	Nonlinear Vibrations of Two-Degree-of-Freedom Gyroscopic Systems, 394	
15.3	Parametrically Excited Gyroscopic Systems, 397	
15.4	Second-Order Differential Systems, 401	
15.5	General Boundary Conditions, 406	
15.6	A Simple Eigenvalue Problem, 412	
15.7	A Degenerate Eigenvalue Problem, 414	
15.8	Acoustic Waves in a Duct with Sinusoidal Walls, 418	
15.9	Vibrations of Nearly Circular Membranes, 426	
15.10	A Fourth-Order Differential System, 432	
15.11	General Fourth-Order Differential Systems, 438	
15.12	A Fourth-Order Eigenvalue Problem, 441	
15.13	A Differential System of Equations, 445	
15.14	General Differential Systems of First-Order Equations, 447	
15.15	Differential Systems with Interfacial Boundary Conditions, 452	

xiv CONTENTS

15.16 Integral Equations, 454

15.17 Partial-Differential Equations, 458

Exercises, 462

APPENDIX A TRIGONOMETRIC IDENTITIES 472

**APPENDIX B LINEAR ORDINARY-DIFFERENTIAL
EQUATIONS 480**

BIBLIOGRAPHY 501

INDEX 507

CHAPTER 1

Introduction

1.1. Dimensional Analysis

Exact solutions are rare in many branches of fluid mechanics, solid mechanics, motion, and physics because of nonlinearities, inhomogeneities, and general boundary conditions. Hence, engineers, physicists, and applied mathematicians are forced to determine approximate solutions of the problems they are facing. These approximations may be purely numerical, purely analytical, or a combination of numerical and analytical techniques. In this book, we concentrate on the purely analytical techniques, which, when combined with a numerical technique such as a finite-difference or a finite-element technique, yield very powerful and versatile techniques.

The key to solving modern problems is mathematical modeling. This process involves keeping certain elements, neglecting some, and approximating yet others. To accomplish this important step, one needs to decide the order of magnitude (i.e., smallness or largeness) of the different elements of the system by comparing them with each other as well as with the basic elements of the system. This process is called *nondimensionalization* or making the variables *dimensionless*. Consequently, one should always introduce dimensionless variables before attempting to make any approximations. For example, if an element has a length of one centimeter, would this element be large or small? One cannot answer this question without knowing the problem being considered. If the problem involves the motion of a satellite in an orbit around the earth, then one centimeter is very very small. On the other hand, if the problem involves intermolecular distances, then one centimeter is very very large. As a second example, is one gram small or large? Again one gram is very very small compared with the mass of a satellite but it is very very large compared with the mass of an electron. Therefore, expressing the equations in dimensionless form brings out the important dimensionless parameters that govern the behavior of the system. Even if one is not interested in approximations, it is recommended that one perform this important step before analyzing the system or presenting

2 INTRODUCTION

experimental data. Next, we give a few examples illustrating the process of nondimensionalization.

EXAMPLE 1

We consider the motion of a particle of mass m restrained by a linear spring having the constant k and a viscous damper having the coefficient μ , as shown in Figure 1-1. Using Newton's second law of motion, we have

$$m \frac{d^2 u}{dt^2} + \mu \frac{du}{dt} + ku = 0 \quad (1.1)$$

where u is the displacement of the particle and t is time. Let us assume that the particle was released from rest from the position u_0 so that the initial conditions are

$$u(0) = u_0 \quad \frac{du}{dt}(0) = 0 \quad (1.2)$$

In this case, u is the dependent variable and t is the independent variable. They need to be made dimensionless by using a characteristic distance and a characteristic time of the system. The displacement u can be made dimensionless by using the initial displacement u_0 as a characteristic distance, whereas the time t can be made dimensionless by using the inverse of the system's natural frequency $\omega_0 = \sqrt{k/m}$. Thus, we put

$$u^* = \frac{u}{u_0} \quad t^* = \omega_0 t$$

where the asterisk denotes dimensionless quantities. Then,

$$\begin{aligned} \frac{du}{dt} &= \frac{d(u_0 u^*)}{dt^*} \cdot \frac{dt^*}{dt} = \omega_0 u_0 \frac{du^*}{dt^*} \\ \frac{d^2 u}{dt^2} &= \omega_0^2 u_0 \frac{d^2 u^*}{dt^{*2}} \end{aligned}$$

so that (1.1) becomes

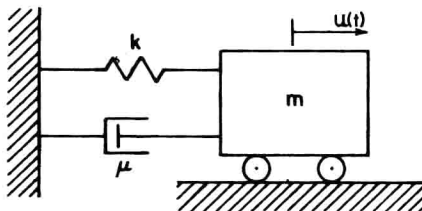


Figure 1-1. A mass restrained by a spring and a viscous damper.

$$m\omega_0^2 u_0 \frac{d^2 u^*}{dt^{*2}} + \mu\omega_0 u_0 \frac{du^*}{dt^*} + ku_0 u^* = 0$$

Hence,

$$\frac{d^2 u^*}{dt^{*2}} + \mu^* \frac{du^*}{dt^*} + \frac{k}{m\omega_0^2} u^* = 0$$

or

$$\frac{d^2 u^*}{dt^{*2}} + \mu^* \frac{du^*}{dt^*} + u^* = 0 \quad (1.3)$$

where

$$\mu^* = \frac{\mu}{m\omega_0} \quad (1.4)$$

In terms of the above dimensionless quantities, (1.2) becomes

$$u^*(0) = 1 \quad \text{and} \quad \frac{du^*}{dt^*}(0) = 0 \quad (1.5)$$

Thus, the solution to the present problem depends only on the single parameter μ^* , which represents the ratio of the damping force to the inertia force or the restoring force of the spring. If this ratio is small, then one can use the dimensionless quantity μ^* as the small parameter in obtaining an approximate solution of the problem, and we speak of a lightly damped system. We should note that the system cannot be considered lightly damped just because μ is small; $\mu^* = \mu/m\omega_0 = \mu/\sqrt{km}$ must be small.

EXAMPLE 2

Let us assume that the spring force is a nonlinear function of u according to

$$f_{\text{spring}} = ku + k_2 u^2 \quad (1.6)$$

where k and k_2 are constants. Then, (1.1) becomes

$$m \frac{d^2 u}{dt^2} + \mu \frac{du}{dt} + ku + k_2 u^2 = 0 \quad (1.7)$$

Again, using the same dimensionless quantities as in the preceding example, we have

$$mu_0\omega_0^2 \frac{d^2 u^*}{dt^{*2}} + \mu u_0\omega_0 \frac{du^*}{dt^*} + ku_0 u^* + k_2 u_0^2 u^{*2} = 0$$

or

4 INTRODUCTION

$$\frac{d^2 u^*}{dt^{*2}} + \mu^* \frac{du^*}{dt^*} + u^* + \epsilon u^{*2} = 0 \quad (1.8)$$

where

$$\mu^* = \frac{\mu}{m\omega_0} \quad \text{and} \quad \epsilon = \frac{k_2 u_0}{k} \quad (1.9)$$

The initial conditions transform as in (1.5). Thus, the present problem is a function of the two dimensionless parameters μ^* and ϵ . As before, μ^* represents the ratio of the damping force to the inertia force or the linear restoring force. The parameter ϵ represents the ratio of the nonlinear and linear restoring forces of the spring.

When we speak of a weakly nonlinear system, we mean that $k_2 u_0/k$ is small. Even if k_2 is small compared with k , the nonlinearity will not be small if u_0 is large compared with k/k_2 . Thus, ϵ is the parameter that characterizes the nonlinearity.

EXAMPLE 3

As a third example, we consider the motion of a spaceship of mass m that is moving in the gravitational field of two fixed mass-centers whose masses m_1 and m_2 are much much bigger than m . With respect to the Cartesian coordinate system shown in Figure 1-2, the equations of motion are

$$m \frac{d^2 x}{dt^2} = - \frac{mm_1 Gx}{(x^2 + y^2)^{3/2}} - \frac{mm_2 G(x-L)}{[(x-L)^2 + y^2]^{3/2}} \quad (1.10)$$

$$m \frac{d^2 y}{dt^2} = - \frac{mm_1 Gy}{(x^2 + y^2)^{3/2}} - \frac{mm_2 Gy}{[(x-L)^2 + y^2]^{3/2}} \quad (1.11)$$

where t is the time, G is the gravitational constant, and L is the distance between m_1 and m_2 .

In this case, the dependent variables are x and y and the independent variable is t . Clearly, a characteristic length of the problem is L , the distance between the two mass centers. A characteristic time of the problem is not as obvious. Since

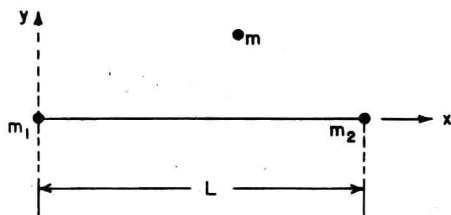


Figure 1-2. A satellite in the gravitational field of two fixed mass centers.

the motions of the masses m_1 and m_2 are assumed to be independent of that of the spaceship, m_1 and m_2 move about their center of mass in ellipses. The period of oscillation is

$$T = \frac{2\pi L^{3/2}}{\sqrt{G(m_1 + m_2)}}$$

so that the frequency of oscillation is

$$\omega_0 = L^{-3/2} \sqrt{G(m_1 + m_2)} \quad (1.12)$$

Thus, we use the inverse of ω_0 as a characteristic time. Then, we introduce dimensionless quantities defined by

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad t^* = \omega_0 t \quad (1.13)$$

so that

$$\begin{aligned} \frac{dx}{dt} &= \frac{d(x^*L)}{dt^*} \frac{dt^*}{dt} = L\omega_0 \frac{dx^*}{dt^*} & \frac{d^2x}{dt^2} &= L\omega_0^2 \frac{d^2x^*}{dt^{*2}} \\ \frac{dy}{dt} &= \frac{d(y^*L)}{dt^*} \frac{dt^*}{dt} = L\omega_0 \frac{dy^*}{dt^*} & \frac{d^2y}{dt^2} &= L\omega_0^2 \frac{d^2y^*}{dt^{*2}} \end{aligned}$$

Hence, (1.10) and (1.11) become

$$mL\omega_0^2 \frac{d^2x^*}{dt^{*2}} = - \frac{mm_1GLx^*}{[L^2(x^{*2} + y^{*2})]^{3/2}} - \frac{mm_2GL(x^* - 1)}{[L^2(x^* - 1)^2 + L^2y^{*2}]^{3/2}}$$

$$mL\omega_0^2 \frac{d^2y^*}{dt^{*2}} = - \frac{mm_1GLy^*}{[L^2(x^{*2} + y^{*2})]^{3/2}} - \frac{mm_2GLy^*}{[L^2(x^* - 1)^2 + L^2y^{*2}]^{3/2}}$$

or

$$\frac{d^2x^*}{dt^{*2}} = - \frac{m_1G}{L^3\omega_0^2} \frac{x^*}{(x^{*2} + y^{*2})^{3/2}} - \frac{m_2G}{L^3\omega_0^2} \frac{(x^* - 1)}{[(x^* - 1)^2 + y^{*2}]^{3/2}} \quad (1.14)$$

$$\frac{d^2y^*}{dt^{*2}} = - \frac{m_1G}{L^3\omega_0^2} \frac{y^*}{(x^{*2} + y^{*2})^{3/2}} - \frac{m_2G}{L^3\omega_0^2} \frac{y^*}{[(x^* - 1)^2 + y^{*2}]^{3/2}} \quad (1.15)$$

Using (1.12), we have

$$\frac{m_1G}{L^3\omega_0^2} = \frac{m_1}{m_1 + m_2} \quad \frac{m_2G}{L^3\omega_0^2} = \frac{m_2}{m_1 + m_2}$$

Hence, if we put

$$\frac{m_2}{m_1 + m_2} = \epsilon \quad \text{then} \quad \frac{m_1}{m_1 + m_2} = 1 - \epsilon \quad (1.16)$$

6 INTRODUCTION

and (1.14) and (1.15) become

$$\frac{d^2 x^*}{dt^{*2}} = -\frac{(1-\epsilon)x^*}{(x^{*2} + y^{*2})^{3/2}} - \frac{\epsilon(x^* - 1)}{[(x^* - 1)^2 + y^{*2}]^{3/2}} \quad (1.17)$$

$$\frac{d^2 y^*}{dt^{*2}} = -\frac{(1-\epsilon)y^*}{(x^{*2} + y^{*2})^{3/2}} - \frac{\epsilon y^*}{[(x^* - 1)^2 + y^{*2}]^{3/2}} \quad (1.18)$$

Therefore, the problem depends only on the parameter ϵ , which is usually called the reduced mass. If m_1 represents the mass of the earth and m_2 the mass of the moon, then

$$\epsilon \approx \frac{\frac{1}{80}}{1 + \frac{1}{80}} = \frac{1}{81}$$

which is small and can be used as a perturbation parameter in determining an approximate solution to the motion of a spacecraft in the gravitational field of the earth and the moon.

EXAMPLE 4

As a fourth example, we consider the vibration of a clamped circular plate of radius a under the influence of a uniform radial load. If w is the transverse displacement of the plate, then the linear vibrations of the plate are governed by

$$D\nabla^4 w - P\nabla^2 w - \rho \frac{\partial^2 w}{\partial t^2} = 0 \quad (1.19)$$

where t is the time, D is the plate rigidity, P is the uniform radial load, and ρ is the plate density per unit area. The boundary conditions are

$$\begin{aligned} w = 0 \quad \frac{\partial w}{\partial r} = 0 \quad \text{at } r = a \\ w < \infty \quad \text{at } r = 0 \end{aligned} \quad (1.20)$$

In this case, w is the dependent variable and t and r are the independent variables. Clearly, a is a characteristic length of the problem. The characteristic time is assumed to be T and it is specified below. Then, we define dimensionless variables according to

$$w^* = \frac{w}{a} \quad r^* = \frac{r}{a} \quad t^* = \frac{t}{T}$$

Hence,