Knots and Feynman Diagrams

CAMBRIDGE LECTURE NOTES IN PHYSICS 13

Dirk Kreimer

This book provides an accessible and up-to-date introduction to how knot theory and Feynman diagrams can be used to illuminate problems in quantum field theory.

Beginning with a summary of key ideas from perturbative quantum field theory and an introduction to the Hopf algebra structure of renormalization, early chapters discuss the rationality of ladder diagrams and simple link diagrams. The necessary basics of knot theory are then presented and the number-theoretic relationship between the topology of Feynman diagrams and knot theory is explored. Later chapters discuss four-term relations motivated by the discovery of Vassiliev invariants in knot theory and draw a link to algebraic structures recently observed in noncommutative geometry. Detailed references are included.

Dealing with material at perhaps the most productive interface between mathematics and physics, the book will not only be of considerable interest to theoretical and particle physicists, but also to many mathematicians.

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DIRK KREIMER obtained a PhD in physics from Mainz University in 1992. He held a post-doctoral position at Mainz University before spending two years doing research at the University of Tasmania. In 1995 he returned to Mainz as a physics lecturer. He has given lectures as a guest scientist in universities throughout the world and has been invited to speak at many major conferences. In 1997 Dr Kreimer was awarded a Heisenberg Fellowship of the German Research Council and in 1999 he was visitor at Harvard and was made a fellow of the Clay Mathematical Institute, Cambridge, Massachusetts.

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für Susanne

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When the ideas of a relation between number theory and knot theory through Feynman diagrams became clearer, he was happy to attend a weekly informal seminar of mine, together with Peter Jarvis and Ioannis Tsohantjis, who were the audience on which some of the ideas in this book were tested first.

I cannot express in proper words the amount of gratitude I feel for the fact that I could gain David Broadhurst as a collaborator at an early stage. David enthusiastically responded to the idea of a connection between knots, numbers and field theory, and when he first heard of it he was determined to see if it could be falsified. As it survived, he made all his computational skills available to explore and push these ideas to the limit.

Since the idea was at first intuitive in nature, it would have not reached the maturity of proper science without testing by concrete calculational results at high loop orders, which would have never been achieved without David.

I continued to be lucky. When, eventually, it occurred to me that there is a Hopf algebra structure disentangling the counterterms of arbitrary Feynman graphs into primitive graphs, one of the first opportunities to present it in public was at the IHES (Bures-sur-Yvette), following an invitation by Alain Connes.

It took Alain and I only a short while to discover that the Hopf algebra in non-commutative geometry (NCG), found by him and Henri Moscovici at just about the same time, was related to mine. This resulted in a wonderful collaboration relating QFT to a beautiful piece of concrete modern mathematics.

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Contents

1
Introduction

1.1 Motivation

This book addresses perturbative Quantum Field Theory (pQFT). This may seem to be an old-fashioned subject. Looking at Schweber's wonderful account of its history [Schweber 1994], it is indeed clear that it goes back more than half a century. But, nevertheless, pQFT has dominated particle physics ever since. It is still the only successful tool we have to calculate everyday cross-sections required to compare our present understanding of theory with experiment. On the other hand, the quest to overcome pQFT has spurred a large amount of the recent developments in high-energy physics theory. This quest stems from the fact that pQFT has several features which are widely regarded as unsatisfactory. Amongst them is the problem of ultraviolet (UV)-divergences. From the very beginning, it was the most prominent problem of pQFT, and its solution was often regarded as technical and as lacking in elegance. The idea to overcome pQFT by something which was not based on seemingly ill-defined quantities from the very beginning is usually accepted as a motivation for other approaches to QFT. Thus, there are two major currents in present day approaches to QFT. On the one hand, there is the inelegant technical machinery of pQFT, pushed forward by the practitioners of multiloop calculations, testing and so far confirming the theory to higher and higher levels in the perturbative loop expansion of the Standard Model of elementary particle physics. On the other hand, at a conceptual level, pQFT is often considered insufficient. Thus, there are alternative approaches, notably string theory, inspired by beautiful mathematics [Dijkgraaf 1997], but unable so far to relate to phenomena as they appear in the laboratory.

It is the hope of the author that this book might reconcile the two ends to some extent. It reports recent developments which concern structures and patterns of remarkable and unexpected beauty in the setup of pQFT. These patterns pop up in the most notorious corner of pQFT: the very presence of ill-defined, UV-singular, integrals.

This book will report on these developments in almost inverted historical order, for conceptual clarity. In 1994, the idea emerged that there might be a connection between low-dimensional topology, renormalization and number theory [Kreimer 1997a]. The novel observation was the fact that certain Feynman diagrams deliver overall counterterms which are Laurent series with coefficients in $\mathbb{Q}$, the number field of rationals [Kreimer 1995, Delbourgo et al. 1995, Delbourgo et al. 1996, Kreimer 1997a]. It turned out that this always happened when the topology of these diagrams was relatively simple, while the concrete realization of these overall counterterms in one or the other pQFT was of lesser importance. The subject began to flourish when, in an intense collaboration between the author and David Broadhurst, it turned out that Feynman diagrams of more complicated topology deliver coefficients $\notin \mathbb{Q}$ [Broadhurst and Kreimer 1995, Broadhurst and Kreimer 1996]. And, even more remarkably, the topological differences of such diagrams could be described by associating various different knots with them, by some empirical and ad hoc rules [Broadhurst and Kreimer 1995, Broadhurst and Kreimer 1996, Kreimer 1997a]. This, in turn, established a faithful knot to number dictionary: whenever a certain graph delivered a certain knot, there was a corresponding transcendental as a coefficient in its overall counterterm.

The underlying philosophy is loosely described as follows: assume you have a way to assign a Laurent series in a parameter ϵ to each Feynman graph. You are interested in the limit $\epsilon \to 0$. The pole terms in this Laurent series reflect the ill-definedness of the Feynman integrals associated with these graphs. You may wonder if you can infer from these pole terms the topology of the graph under consideration. This would mean that your mapping from graphs to numbers – coefficients in your Laurent series – is a topological invariant, in the sense that topologically distinct graphs evaluate to different numbers. But what do we mean by saying, that these numbers differ? Let us say that two numbers differ if they are not rational multiples of each other. This suggests