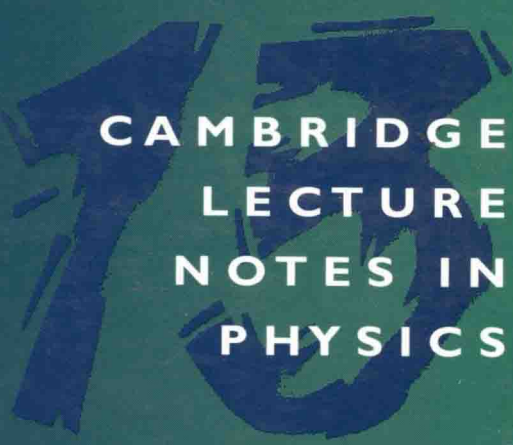


Knots and Feynman Diagrams



CAMBRIDGE
LECTURE
NOTES IN
PHYSICS

Dirk Kreimer

This book provides an accessible and up-to-date introduction to how knot theory and Feynman diagrams can be used to illuminate problems in quantum field theory.

Beginning with a summary of key ideas from perturbative quantum field theory and an introduction to the Hopf algebra structure of renormalization, early chapters discuss the rationality of ladder diagrams and simple link diagrams. The necessary basics of knot theory are then presented and the number-theoretic relationship between the topology of Feynman diagrams and knot theory is explored. Later chapters discuss four-term relations motivated by the discovery of Vassiliev invariants in knot theory and draw a link to algebraic structures recently observed in noncommutative geometry. Detailed references are included.

Dealing with material at perhaps the most productive interface between mathematics and physics, the book will not only be of considerable interest to theoretical and particle physicists, but also to many mathematicians.

Cambridge Lecture Notes in Physics

Editors

PETER GODDARD, *University of Cambridge*

JULIA YEOMANS, *University of Oxford*

This series provides a vehicle for the publication of informal lecture note volumes in all areas of theoretical and experimental physics for graduate students and researchers. It covers condensed matter physics, plasma physics, optics, atomic and molecular physics, particle physics, nuclear physics, field theory, relativity and cosmology, astrophysics, statistical physics, nonlinear dynamics and chaos, and mathematical physics.

CAMBRIDGE
UNIVERSITY PRESS

ISBN 0-521-58761-1



9 780521 587617

KRUMHOLTZ Knots and Feynman Diagrams



Knots and Feynman Diagrams

DIRK KREIMER

Mainz University



CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK <http://www.cup.cam.ac.uk>
40 West 20th Street, New York, NY 10011-4211, USA <http://www.cup.org>
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

© Dirk Kreimer 2000

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2000

Printed in the United Kingdom at the University Press, Cambridge

Typeset by the Author [CRC]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN 0 521 58761 1 paperback

Knots and Feynman Diagrams

This book provides an accessible and up-to-date introduction to how knot theory and Feynman diagrams can be used to illuminate problems in quantum field theory.

Beginning with a summary of key ideas from perturbative quantum field theory and an introduction to the Hopf algebra structure of renormalization, early chapters discuss the rationality of ladder diagrams and simple link diagrams. The necessary basics of knot theory are then presented and the number-theoretic relationship between the topology of Feynman diagrams and knot theory is explored. Later chapters discuss four-term relations motivated by the discovery of Vassiliev invariants in knot theory and draw a link to algebraic structures recently observed in non-commutative geometry. Detailed references are included.

Dealing with material at perhaps the most productive interface between mathematics and physics, the book will not only be of considerable interest to theoretical and particle physicists, but also to many mathematicians.

DIRK KREIMER obtained a PhD in physics from Mainz University in 1992. He held a post-doctoral position at Mainz University before spending two years doing research at the University of Tasmania. In 1995 he returned to Mainz as a physics lecturer. He has given lectures as a guest scientist in universities throughout the world and has been invited to speak at many major conferences. In 1997 Dr Kreimer was awarded a Heisenberg Fellowship of the German Research Council and in 1999 he was visitor at Harvard and was made a fellow of the Clay Mathematical Institute, Cambridge, Massachusetts.

CAMBRIDGE LECTURE NOTES IN PHYSICS 13

General Editors: P. Goddard, J. Yeomans

1. Clarke: The Analysis of Space-Time Singularities
2. Dorey: Exact s-Matrices in Two Dimensional Quantum Field Theory
3. Sciama: Modern Cosmology and the Dark Matter Problem
4. Veltman: Diagrammatica – The Path to Feynman Rules
5. Cardy: Scaling and Renormalization in Statistical Physics
6. Heusler: Black Hole Uniqueness Theorems
7. Coles and Ellis: Is the Universe Open or Closed?
8. Razumov and Saveliev: Lie Algebras, Geometry, and Toda-type Systems
9. Forshaw and Ross: Quantum Chromodynamics and the Pomeron
10. Jensen: Self-organised Criticality
11. Vanderzande: Lattice Models of Polymers
12. Esposito: Dirac Operators and Spectral Geometry
13. Kreimer: Knots and Feynman Diagrams
14. Dorfman: An introduction to Chaos in Nonequilibrium Statistical Mechanics

für Susanne

Acknowledgements

Thanks are due to many friends and colleagues. When, in 1993, I went to Tasmania to do my first postdoc with Bob Delbourgo, it became clear quite quickly that this was a very fortunate choice. Most generously, Bob would let me follow my own ideas, offering me his valuable advice and support whenever needed.

When the ideas of a relation between number theory and knot theory through Feynman diagrams became clearer, he was happy to attend a weekly informal seminar of mine, together with Peter Jarvis and Ioannis Tsohantjis, who were the audience on which some of the ideas in this book were tested first.

I cannot express in proper words the amount of gratitude I feel for the fact that I could gain David Broadhurst as a collaborator at an early stage. David enthusiastically responded to the idea of a connection between knots, numbers and field theory, and when he first heard of it he was determined to see if it could be falsified. As it survived, he made all his computational skills available to explore and push these ideas to the limit.

Since the idea was at first intuitive in nature, it would have not reached the maturity of proper science without testing by concrete calculational results at high loop orders, which would have never been achieved without David.

I continued to be lucky. When, eventually, it occurred to me that there is a Hopf algebra structure disentangling the counterterms of arbitrary Feynman graphs into primitive graphs, one of the first opportunities to present it in public was at the IHES (Bures-sur-Yvette), following an invitation by Alain Connes.

It took Alain and I only a short while to discover that the Hopf algebra in non-commutative geometry (NCG), found by him and Henri Moscovici at just about the same time, was related to mine. This resulted in a wonderful collaboration relating QFT to

a beautiful piece of concrete modern mathematics.

To Bob, David and Alain I owe the greatest debt, for their friendship, support and collaboration.

Many other people contributed to this book, through stimulating discussions and exchange of ideas. Amongst them are John Gracey, whose collaboration was crucial in understanding double sums in field theory, Lou Kauffman, who patiently introduced me to knot theory personally and through his books, Pierre Cartier, who invited me to participate in his seminar on MZVs, as well as Sasha Goncharov, Arthur Jaffe, Roger Picken, Ivan Todorov and Don Zagier.

Also, I owe many thanks to Andrei Davydychev, Volodya Smirnov and other friends and colleagues from the ‘multi-loop fraternity’, whose concerns and interests provided the background for the material of this book.

Christian Bauer, Isabella Bierenbaum, Alexander Frink, Richard Kreckel, Mathias Mertens and Oliver Welzel where a stimulating crowd of students in lectures on the topic of this book.

Arthur Greenspoon helped smoothing the manuscript.

I thank the physics departments at University of Tasmania and Mainz University, as well as the IHES and the Clay Mathematics Institute, for hospitality and partial support during times when this book was written.

Further, I thank the German Research Council (DFG) for support by a Heisenberg Fellowship. Mobility in the last four years was supported by HUCAM grant CHRX-CT94-0579.

Contents

	Acknowledgements	page xi
1	Introduction	1
1.1	Motivation	1
2	Perturbative quantum field theory	7
2.1	pQFT	7
	2.1.1 <i>Canonical quantization</i>	7
	2.1.2 <i>Perturbation theory</i>	10
	2.1.3 <i>Feynman rules</i>	15
	2.1.4 <i>Schwinger–Dyson equations</i>	17
2.2	Regularization	21
	2.2.1 <i>Dimensional regularization</i>	23
2.3	Basic facts about renormalization	25
	2.3.1 <i>Multiplicative renormalization</i>	26
	2.3.2 <i>Power counting</i>	29
3	The Hopf algebra structure of renormalization	36
3.1	Preliminaries	36
3.2	Vertex corrections	41
	3.2.1 <i>The first iteration</i>	41
	3.2.2 <i>The factorization</i>	44
3.3	Overlapping divergences	48
	3.3.1 <i>A toy model</i>	49
3.4	Technicalities	52
	3.4.1 <i>Form factors</i>	53
	3.4.2 <i>Other degrees of divergence</i>	54
3.5	Towards a Hopf algebra	56
3.6	Feynman diagrams as a realization	61
3.7	The Hopf algebra	67
	3.7.1 <i>The coproduct</i>	72
	3.7.2 <i>The antipode</i>	75

3.8	Realizations of \mathcal{A}	80
3.8.1	<i>Toy models</i>	80
3.8.2	<i>Quantum field theories</i>	87
3.8.3	<i>Once more: overlapping divergences</i>	90
3.9	An ultimate example	93
3.10	Remarks	94
4	Rationality: no knots, no transcendentals	97
4.1	A combinatorial approach	99
4.2	Delbourgo's argument	102
4.3	Toy models versus QFT	103
5	The simplest link diagrams	106
5.1	Link diagrams from ladder diagrams	106
5.1.1	<i>Disentangling the link diagram</i>	108
5.1.2	<i>Gauss codes</i>	112
5.2	Links and ladders – the overlapping case	114
6	Necessary topics from knot theory	118
6.1	Basics	118
6.2	Torus knots	121
6.3	Braids	123
6.4	Knot polynomials	125
7	Knots to numbers: $(2, 2n - 3)$ torus knots and $\zeta(2n - 3)$	130
7.1	$\zeta(3)$ from a counterterm	130
7.2	The $(2, q)$ torus knots and $\zeta(q)$	134
7.3	Factor knots	139
7.4	Gauss codes	141
8	One-loop words	143
8.1	Definitions	144
8.2	An elementary example	147
8.3	Continuation to a dressed two-loop graph	154
8.4	Higher order dressing	160
9	Euler–Zagier sums	163
9.1	Relations coming from the Drinfeld associator	165
9.2	Shuffle algebras	168
9.3	Euler–Zagier sums and MZVs	169

10	Knots and transcendentals	174
10.1	The (3, 4) torus knot and the first Euler double sum	176
	10.1.1 <i>The knot 8_{19} and its knot-number</i>	177
	10.1.2 <i>A momentum routing analysis</i>	179
	10.1.3 <i>Gauss code analysis</i>	182
	10.1.4 <i>Chord diagrams, knots and numbers to five loops</i>	183
10.2	ϕ^4 -theory: more knots and numbers	183
10.3	Rationality and the β -function of quenched QED	189
10.4	Euler double sums	193
10.5	Field theory, knot theory, number theory	196
10.6	From knots to numbers	199
	10.6.1 <i>How many knot numbers?</i>	200
	10.6.2 <i>Knot-numbers from evaluations of Feynman diagrams</i>	201
	10.6.3 <i>Positive knots associated with irreducible MZVs</i>	205
11	The four-term relation	214
11.1	Introduction	214
11.2	The 4TR between primitive graphs	218
11.3	A test	232
11.4	Hopf algebra and 4TR	234
12	Hopf algebras, non-commutative geometry, and what else?	236
12.1	The Hopf algebra \mathcal{H}_R of rooted trees	236
	12.1.1 <i>Formulae for renormalization</i>	241
12.2	The relation between \mathcal{H}_R and \mathcal{H}_T	243
12.3	And what else?	247
	References	253
	Index	259

Introduction

1.1 Motivation

This book addresses perturbative Quantum Field Theory (pQFT). This may seem to be an old-fashioned subject. Looking at Schweber's wonderful account of its history [Schweber 1994], it is indeed clear that it goes back more than half a century. But, nevertheless, pQFT has dominated particle physics ever since. It is still the only successful tool we have to calculate everyday cross-sections required to compare our present understanding of theory with experiment. On the other hand, the quest to overcome pQFT has spurred a large amount of the recent developments in high-energy physics theory. This quest stems from the fact that pQFT has several features which are widely regarded as unsatisfactory. Amongst them is the problem of ultraviolet (UV)-divergences. From the very beginning, it was the most prominent problem of pQFT, and its solution was often regarded as technical and as lacking in elegance. The idea to overcome pQFT by something which was not based on seemingly ill-defined quantities from the very beginning is usually accepted as a motivation for other approaches to QFT. Thus, there are two major currents in present day approaches to QFT. On the one hand, there is the inelegant technical machinery of pQFT, pushed forward by the practitioners of multiloop calculations, testing and so far confirming the theory to higher and higher levels in the perturbative loop expansion of the Standard Model of elementary particle physics. On the other hand, at a conceptual level, pQFT is often considered insufficient. Thus, there are alternative approaches, notably string theory, inspired by beautiful mathematics [Dijkgraaf 1997], but unable so far to relate to phenomena as they appear in the laboratory.

It is the hope of the author that this book might reconcile the two ends to some extent. It reports recent developments which

concern structures and patterns of remarkable and unexpected beauty in the setup of pQFT. These patterns pop up in the most notorious corner of pQFT: the very presence of ill-defined, UV-singular, integrals.

This book will report on these developments in almost inverted historical order, for conceptual clarity. In 1994, the idea emerged that there might be a connection between low-dimensional topology, renormalization and number theory [Kreimer 1997a]. The novel observation was the fact that certain Feynman diagrams deliver overall counterterms which are Laurent series with coefficients in \mathbb{Q} , the number field of rationals [Kreimer 1995, Delbourgo et al. 1995, Delbourgo et al. 1996, Kreimer 1997a]. It turned out that this always happened when the topology of these diagrams was relatively simple, while the concrete realization of these overall counterterms in one or the other pQFT was of lesser importance. The subject began to flourish when, in an intense collaboration between the author and David Broadhurst, it turned out that Feynman diagrams of more complicated topology deliver coefficients $\notin \mathbb{Q}$ [Broadhurst and Kreimer 1995, Broadhurst and Kreimer 1996]. And, even more remarkably, the topological differences of such diagrams could be described by associating various different knots with them, by some empirical and ad hoc rules [Broadhurst and Kreimer 1995, Broadhurst and Kreimer 1996, Kreimer 1997a]. This, in turn, established a faithful knot to number dictionary: whenever a certain graph delivered a certain knot, there was a corresponding transcendental as a coefficient in its overall counterterm.

The underlying philosophy is loosely described as follows: assume you have a way to assign a Laurent series in a parameter ϵ to each Feynman graph. You are interested in the limit $\epsilon \rightarrow 0$. The pole terms in this Laurent series reflect the ill-definedness of the Feynman integrals associated with these graphs. You may wonder if you can infer from these pole terms the topology of the graph under consideration. This would mean that your mapping from graphs to numbers – coefficients in your Laurent series – is a topological invariant, in the sense that topologically distinct graphs evaluate to different numbers. But what do we mean by saying, that these numbers differ? Let us say that two numbers differ if they are not rational multiples of each other. This suggests