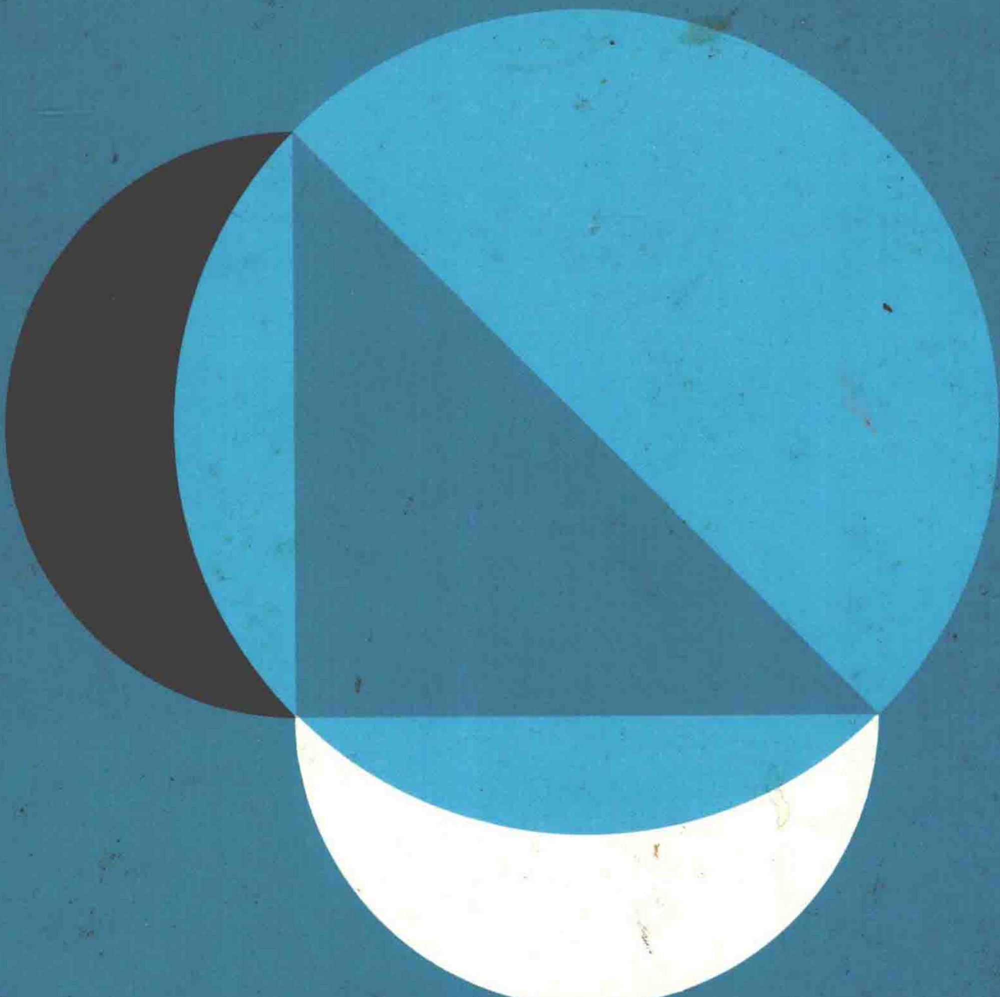
GEOMETRY

Moise & Downs



GEOMETRY

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CONSULTING EDITOR

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PREFACE

This edition of *Geometry* is the first major revision of this text. After six years, it is hard to remember that when the book first appeared it was regarded as boldly innovative, and many of its novelties were regarded with misgivings. This applies in particular to the logical accuracy of its language, to the completeness of its proofs, and to the use of explicit notations to distinguish lines, rays, segments, and lengths. It turned out in practice that these features were just as good pedagogically as they were logically; and our notation is now standard in nearly all widely used geometry books. Since these features were successful, there was obviously no need to change them.

Nevertheless, in the past six years the world has moved, and in geometry it has moved faster than usual. The first edition was intended for students who had not previously had a modern mathematics course. Consequently, the authors were led to take precautions which are no longer needed. Nearly all of the present generation of students have heard of equivalence relations before they take a course in geometry, and the fact is that congruence is an equivalence relation. Therefore we now say, matter-of-factly, that congruence is an equivalence relation. Similarly, in the first edition we used a variety of functions, but did not call them functions, for fear of using too many new words. Since the word "function" is now familiar, there is no reason to avoid it. At many points we have made such changes, in response to the change in the backgrounds of the students.

The problems in the first edition were carefully designed to be ample for almost every purpose. In the present edition, almost all these problems have been retained. However, many new problems, especially of an easier nature, have been inserted. Also, the layout of the problems has been arranged in such a way that it is easy to tell at a glance which problems are easy, moderate, or difficult. The easy set may often be used as oral exercises. As before, no student is expected to do every problem, and teachers still need to select problems discriminately in assigning homework. Extensive teaching experience with the first edition has prompted us to make slight revisions in the order of topics within the first few chapters, particularly to ease the transition from reading proofs to writing them.

Finally, we have added two new chapters, one on trigonometry and another on transformations and vectors. It seemed very desirable to introduce analytic trigonometry, using directed angles of any size. The transition from purely geometric ideas to analytic ideas would come soon in any case, and it seemed better for this transition to occur within the geometry course. Undirected angles are useful for some purposes (namely, those of elementary geometry), and directed angles are useful for other purposes—in analytic geometry and calculus. Making this point clear is a minor advantage of teaching analytic trigonometry in a geometry course. The major advantage, of course, is that the material is vitally important in later study.

The case for geometric transformations is not quite so strong, because this material is not going to be needed immediately. Nevertheless, the idea of a transformation is central in modern mathematics, and probably the easiest introduction to it is in geometry. Vector ideas are also prominent in mathematics and in the other sciences, and it would seem artificial to leave them unmentioned. Transformations and symmetry naturally go together; the idea of symmetry clarifies much of the geometry that has been studied earlier. In the new Chapter 18, we have tied all these things together.

The treatment of transformations and vectors, in Chapter 18, is not to be confused with various proposals which have been widely discussed in recent years. Some have advocated that transformations be used from the outset, and that all the geometric ideas, including congruence, be formulated in terms of them. We have not carried out any such scheme in the present book, and we believe that such a scheme involves serious difficulties. In spite of its logical simplicity, as judged by mathematicians, it would be far more sophisticated than the present treatment. There is also the difficulty of providing suitable problems. One of the best features of traditional geometry is its wealth of interesting problems that the student can solve. The merits of a teaching plan depend crucially on the sort of problem work that the student is given to do. If some of the problems are dull, and the rest are hopelessly difficult, then the course is not going to work in the classroom, no matter how beautiful it may look in theory. The same considerations apply to a course in which everything is stated, from the beginning, in terms of vectors.

The central purpose of a textbook ought to be to provide the student with a stimulating and attractive life to lead, when he does the work assigned. At least, this is the purpose of the present authors.

The authors would be remiss, indeed, if we failed to acknowledge the assistance of the staff at Addison-Wesley and the criticisms and contributions of the hundreds of teachers and students who have communicated their thoughts to us.

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Edwin E. Moise
Floyd L. Downs, Jr.

Illustration Credits p. 77, Courtesy British Museum; p. 300, Map adapted from Geographia, an ancient manuscript (225 B.C.). The dimensions of the inhabited world are given as: length from ocean to ocean—7780 miles, width—3800 miles; p. 644, Archimedes (Smith Collection, Library of Columbia University).

CHAPTER 1

Common Sense and Exact Reasoning

Objectives . . .

- Understand the difference between geometrical reasoning and geometrical facts.
- Realize that intuition is not always trustworthy.
- Develop concepts of point, line, and plane.

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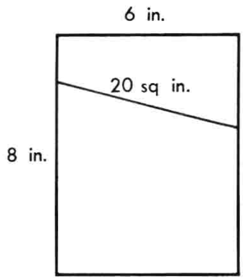
1–1 TWO KINDS OF PROBLEMS

Consider the following problems.

(1) A certain rectangle measures 6 in. by 8 in. The area enclosed is cut into two pieces by a line segment. If the area of one piece is 20 square inches (sq in.), what is the area of the other piece?



(2) In a certain rectangle, the sum of the length and the width is 14 (measured in inches). A second rectangle is five times as long as the first, and three times as wide. The perimeter of the second rectangle is 91. What are the dimensions of the first rectangle?

You should be able to get the answer to Problem 1 without having to think very hard. The answer is 28 sq in., because $6 \cdot 8 = 48$ and $48 - 20 = 28$. Of course, we could solve the problem algebraically, if we wanted to, by setting up the equation

$$20 + x = 6 \cdot 8$$

and then solving to get $x = 28$. But this seems a little silly because it is so unnecessary. The chances are that you solved more difficult problems than this, with arithmetic, before you studied algebra at all. And if all algebraic equations were as superfluous as the one we have just set up, then no serious-minded person would pay any attention to them.

Problem 2, however, is quite another matter. If we denote the length and width of the first rectangle by x and y, then the length and width of the second rectangle are $5x$ and $3y$. Therefore

$$5x + 3y = \tfrac{91}{2},$$

because the sum of the length and width is half of the perimeter. We also know that

$$x + y = 14.$$

This gives us a system of two equations in two unknowns. To solve, we multiply each term in the second equation by 3, getting

$$3x + 3y = 42,$$

and then we subtract this last equation, term by term, from the first. This yields

$$2x = 45\tfrac{1}{2} - 42 = 3\tfrac{1}{2} = \tfrac{7}{2}$$

or

$$x = \tfrac{7}{4} = 1\tfrac{3}{4}.$$

Therefore

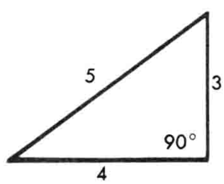
$$y = 14 - 1\tfrac{3}{4} = 12\tfrac{1}{4}.$$

It is now rather easy to check that our answer satisfies the conditions of the problem.

In a way, these two problems are rather similar, but in a very important way, they are quite different. The first is what you might call a common-sense problem. It is easy to guess what the answer ought to be, and it is also easy to check that the natural guess is also the right answer. On the other hand, guessing the answer to the second problem is almost out of the question. To solve it, we need to know something about mathematical methods.

There are cases of this kind in geometry. Consider the following statements.

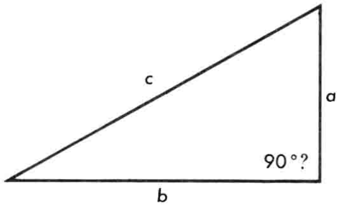
(1) If a triangle has sides of length 3, 4, and 5, then it is a right triangle, with a right angle opposite the longest side.



(2) Let a triangle be given, with sides of length a, b, and c. If

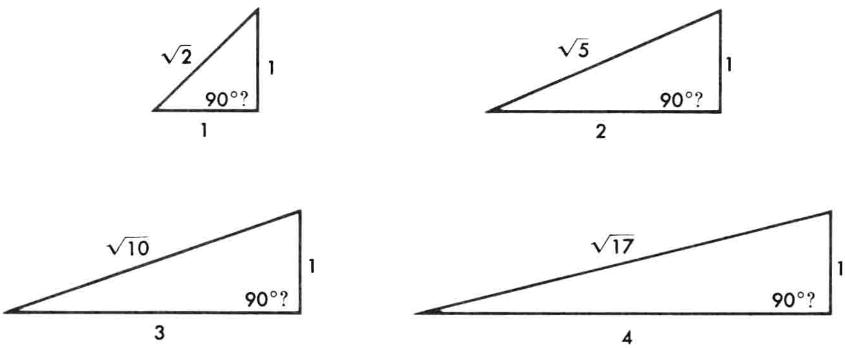
$$a^2 + b^2 = c^2,$$

then the triangle is a right triangle, with a right angle opposite the longest side.



Statement 1 was known to the ancient Egyptians. They checked it by experiment. You can check it yourself by drawing a 3-4-5 triangle, as exactly as possible, and then measuring the angle opposite the longest side with a protractor. You should bear in mind, of course, that this check is only approximate. Suppose, for example, that the angle is really 89° 59′ 59½″ (that is, 89 degrees, 59 minutes, and 59½ seconds) instead of 90° 0′ 0″ exactly. In this case, you would hardly expect to tell the difference with a protractor, no matter how carefully you sharpened your pencil and drew your figure. Nevertheless, the "Egyptian method" is a sound common-sense method of checking an experimental fact.

Statement 2 was not known to the Egyptians. It was discovered much later by the Greeks, and is called the Pythagorean relation. It is impossible to check Statement 2 by experiment because there are infinitely many cases to consider. For instance, you would have to construct triangles, and take readings with a protractor, for all the following cases:



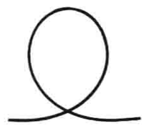
and so on endlessly. Thus it is hopeless to verify our general statement by experiment, even approximately. Therefore, a reasonable person would not be convinced that Statement 2 is true in all cases until he had seen a logical reason why it *has* to be true in all cases.

The Egyptians were extremely skillful at making physical measurements. For example, the edges of the base of the Great Pyramid of Gizeh are about 756 feet (ft) long, and the lengths of these four edges agree, with an error of only about two-thirds of an inch. Nobody seems to know, today, how the builders achieved such accuracy.

But the Greeks discovered a new method which was vastly more powerful. This was the method of exact geometrical reasoning. By this method they turned plausible guesses into solid knowledge, and they learned some rather startling things that nobody would have believed without seeing them proved. In this way, the Greeks laid the foundation for modern mathematics and hence for modern science in general.

Problem Set 1–1

1. How good are you at guessing? Try this experiment. Take a piece of string about 5 ft long, and place it on the floor in a loop with the ends free:



Then pull the ends of the string, gradually making the loop smaller, and stop when you think the loop is the size of your waist. Mark the string where it crosses itself, and check your guess by putting the string around your waist. *After* you have made this check, read the remarks on Problem 1 at the end of this set of problems.

2. This is also a problem in guessing.

A page of a newspaper is not very thick, about 0.003 in., and you often have seen a stack of newspapers. Suppose you were to place one sheet of newspaper on the floor. Next you place another sheet on top of the first, then two more sheets, then four, and so on, building up a pile of newspaper. Each time you add to the pile as many sheets as are already there. After the tenth time you would have a pile about 3 in. high. If you were able to continue until you had added to the pile for the fiftieth time, how high would the pile be?

One of the answers (a) through (d) below is the correct one. All you have to do is guess, or calculate, which one it is.

(a) About as high as your classroom

(b) About as high as a four-story building

(c) About as high as the Empire State Building

(d) More than twice as high as the Empire State Building

After you have made your choice, read the remarks on Problem 2 at the end of this set of problems.

3. The first of the pair of questions below can be answered by "common sense." State only its answer. The second requires some arithmetic or algebraic process for its solution. Show your work for it.

(a) What is one-sixth of 12?

(b) What is one-sixth of 5,255,622?

4. Follow the directions for Problem 3.

(a) One-third of the distance between two cities is 10 miles (mi). What is the distance between them?

(b) The distance between two cities is 10 mi more than one-third of the distance between them. What is the distance between them?

5. An important part of learning mathematics is learning to recognize patterns which suggest general truths. For example, looking at the statements,

$$3 + 5 = 8, \qquad 9 + 5 = 14, \qquad 11 + 17 = 28,$$

you might guess that the sum of two odd numbers is an even number. Can you think of two odd numbers whose sum is an odd number? Does your answer prove that two such odd numbers do not exist?

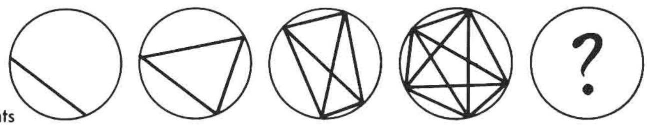
6. Consider these statements:

$$1 + 3 + 5 = 9, \qquad 5 + 7 + 11 = 23, \qquad 9 + 13 + 21 = 43.$$

(a) Complete this generalization: If a, b, and c are three odd numbers, then $a + b + c$ is ____________ .

(b) Can you establish the truth of this generalization?

7. Consider the following figures and the pattern suggested.



Number of points connected:	2	3	4	5	6
Number of regions formed:	2	4	8	16	?

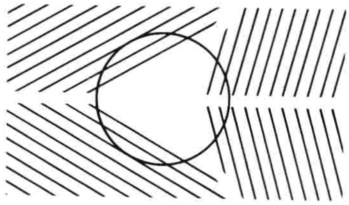
(a) Replace the question mark under the 6 by the number you think belongs there.

(b) Draw a circle and connect any six points on it in all possible ways. Count the regions thus formed. Does the result agree with your answer to part (a)?

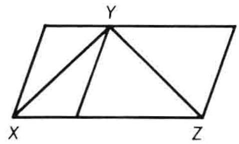
(c) What does this problem indicate about showing whether a generalization is true or false?

8. The following optical illusions show that you cannot always trust appearances to decide upon a fact.

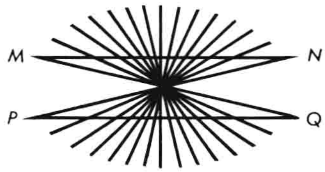
(a) Is this a circle? Test your answer with a dime.



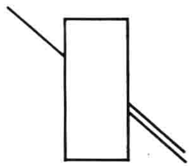
(b) Are XY and YZ the same length? Compare the lengths with your ruler or compass.



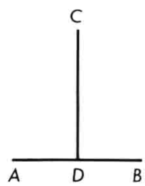
(c) Are MN and PQ straight line segments?



(d) Which line at the right of the rectangle is the continuation of the line at the left?



(e) Which is longer, AB or CD?



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9. If two students carefully and independently measure the width of a classroom with rulers, one measuring from left to right and the other from right to left, they will probably get different results. Try it! Which of the following is a plausible reason for the difference?

(a) The rulers have different lengths.

(b) Things are longer (or shorter) from left to right than from right to left.

(c) The discrepancies resulting from the change in the position of the ruler accumulate, and the sum of these small errors makes a discernible difference.

(d) One person may have lost count.