Pseudolinear Functions *and* Optimization

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To
Our Beloved Parents
Smt. Shyama Mishra, Shri Gauri Shankar Mishra
and
Smt. Urmila Upadhyay, Shri Yadvendra D. Upadhyay

Foreword

Optimization is a discipline that plays a key role in modeling and solving problems in most areas of engineering and sciences. Optimization tools are used to understand the dynamics of information networks, financial markets, solve logistics and supply chain problems, design new drugs, solve biomedical problems, design renewable and sustainable energy systems, reduce pollution, and improve health care systems. In general, we have convex optimization models, which, at least in theory, can be solved efficiently, and nonconvex optimization models that are computationally very hard to solve. Designing efficient algorithms and heuristics for solving optimization problems requires a very good understanding of the mathematics of optimization. For example, the development of optimality conditions, the theory of convexity, and computational complexity theory have been instrumental for the development of computational optimization algorithms.

The book *Pseudolinear Functions and Optimization* by Shashi Kant Mishra and Balendu Bhooshan Upadhyay, sets the mathematical foundations for a class of optimization problems. Although convexity theory plays the most important role in optimization, several attempts have been made to extend the theory to generalized convexity. Such extensions were necessary to understand optimization models in a wide spectrum of applications.

The book *Pseudolinear Functions and Optimization* is to my knowledge the first book that is dedicated to a specific class of generalized convex functions that are called pseudolinear functions. This is an in-depth study of the mathematics of pseudolinear functions and their applications. Most of the recent results on pseudolinear functions are covered in this book.

The writing is pleasant and rigorous and the presentation of material is very clear. The book will definitely will be useful for the optimization community and will have a lasting effect.

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Preface

In 1967 Kortanek and Evans [149] studied the properties of the class of functions, which are both pseudoconvex and pseudoconcave. This class of functions were later termed as pseudolinear functions. In 1984 Chew and Choo [47] derived first and second order characterizations for pseudolinear functions. The linear and quadratic fractional functions are particular cases of pseudolinear functions. Several authors have studied pseudolinear functions and their characterizations, see Cambini and Carosi [34], Schaible and Ibaraki [246], Rapcsak [233], Komlosi [147], Kaul *et al.* [139], Lu and Zhu [171], Dinh *et al.* [67], Zhao and Tang [298], Ansari and Rezaei [4] and Mishra *et al.* [200].

Chapter 1 is introductory and contains basic definitions and concepts needed in the book.

Chapter 2, presents basic properties and characterization results on pseudolinear functions. Further, it includes semilocal pseudolinear functions, Dini differentiable pseudolinear functions, locally Lipschitz pseudolinear functions, h-pseudolinear functions, directionally differentiable pseudolinear functions, weakly pseudolinear functions and their characterizations.

Chapter 3, presents characterizations of solution sets of pseudolinear optimization problems, linear fractional optimization problems, directionally differentiable pseudolinear optimization problems, h-pseudolinear optimization problems and locally Lipschitz optimization problems.

Chapter 4, presents characterizations of solution sets in terms of Lagrange multipliers for pseudolinear optimization problems and its other generalizations given in Chapter 3.

Chapter 5, considers multiobjective pseudolinear optimization problems and multiobjective fractional pseudolinear optimization problems and presents optimality conditions and duality results for these two problems.

Chapter 6, extends the results of Chapter 5 to locally Lipschitz functions using the Clarke subdifferentials.

Chapter 7, considers static minmax pseudolinear optimization problems and static minmax fractional pseudolinear optimization problems and presents optimality conditions and duality results for these two problems.

Chapter 8, extends the results of Chapter 7 to locally Lipschitz functions using the Clarke subdifferentials.

Chapter 9, presents optimality and duality results for h-pseudolinear optimization problems.

Chapter 10, presents optimality and duality results for semi-infinite pseudolinear optimization problems.

Chapter 11, presents relationships between vector variational inequalities and vector optimization problems involving pseudolinear functions. Moreover, relationships between vector variational inequalities and vector optimization problems involving locally Lipschitz pseudolinear functions using the Clarke subdifferentials are also presented.

Chapter 12, presents an extension of pseudolinear functions are used to establish results on variational inequality problems.

Chapter 13, presents results on η-pseudolinear functions and characterizations of solution sets of η-pseudolinear optimization problems.

Chapter 14, presents pseudolinear functions on Riemannian manifolds and characterizations of solution sets of pseudolinear optimization problems on Riemannian manifolds. Moreover, η-pseudolinear functions and characterizations of solution sets of η-pseudolinear optimization problems on differentiable manifolds are also presented.

Chapter 15, presents results on pseudolinearity of quadratic fractional functions.

Chapter 16, extends the class of pseudolinear functions and η-pseudolinear functions to pseudolinear and η-pseudolinear fuzzy mappings and characterizations of solution sets of pseudolinear fuzzy optimization problems and η-pseudolinear fuzzy optimization problems.

Finally, in Chapter 17, some applications of pseudolinear optimization problems to hospital management and economics are given.

The authors are thankful to Prof. Nicolas Hadjisavvas for his help and discussion in Chapter 4. The authors are indebted to Prof. Juan Enrich Martinez-Legaz, Prof. Pierre Marechal, Prof. Dinh The Luc, Prof. Le Thi Hoai An, Prof. Sy-Ming Guu, Prof. King Keung Lai, and Prof. S.K. Neogy for their help, support, and encouragement in the course of writing this book. The authors are also thankful to Ms. Aastha Sharma from CRC Press for her patience and effort in handling the book.

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Balendu Bhooshan Upadhyay

Symbol Description

$:=$	equal to by definition
ϕ	empty set
$\forall$	for all
∞	infinity
$\langle .,. \rangle$	Euclidean inner product
$\|.\|$	Euclidean norm
$\exists$	there exists
0	zero element in the vector space $\mathbb{R}^n$
2^X	family of all subsets of a set X
A^T	transpose of a matrix A
$\text{lin}(A)$	linear hull of a set A
$\text{aff}(A)$	affine hull of a set A
$\arg\min f$	set of all minima of a function f
$\mathbb{B}_r(x)$	open ball with center at x and radius r
$\mathbb{B}_r[x]$	closed ball with center at x and radius r
$\mathbb{B}$	open unit ball
$[x,y]$	closed line segment joining x and y
$]x,y[$	open line segment joining x and y
$\text{bd}(A)$	boundary of a set A
$\text{cl}(A)$	closure of a set A
$\text{co}(A)$	convex hull of a set A
$\text{con}(A)$	conic hull of a set A
$d(x,y)$	distance between x and y
$\text{d}(f)$	domain of a function f
$\text{dom}(f)$	effective domain of map f
$\text{epi}(f)$	epigraph of a function f
$\nabla f(x)$	gradient of a function f at x
$\nabla^2 f(x)$	Hessian matrix of a function f at x
$f'(x;d)$	directional derivative of f at x in the direction d
$f'_+(x;d)$	right sided directional derivative of f at x in the direction d
$f'_-(x;d)$	left sided directional derivative of f at x in the direction d
$f^\circ(x;d)$	Clarke directional derivative of f at x in the direction d
$D^+f(x;d)$	Dini upper directional derivative of f at x in the direction d
$D_+f(x;d)$	Dini lower directional derivative of f at x in the direction d
$f^{DH}(x;d)$	Dini-Hadamard upper directional derivative of f at x in the direction d
$f_{DH}(x;d)$	Dini-Hadamard lower directional derivative of f at x in the direction d
$\partial f(x)$	subdifferential of a function f at x
$\text{supp } u$	support of fuzzy set u
$[u]_\alpha$	$\alpha-$cut set of fuzzy set u
$\mathfrak{I}_0$	family of fuzzy numbers
$\partial^c f(x)$	Clarke subdifferential of a function f at x
$d_C(.)$	distance function of a set C
$\text{graph}(f)$	graph of a function f
$\text{hyp }(f)$	hypograph of a function f
$\delta(.\|C)$	indicator function of a set C
$X^\perp$	orthogonal complement of X
H^+	upper closed half-space
H^{++}	upper open half-space
H^-	lower closed half-space
H^{--}	lower open half-space
$\text{int}(A)$	interior of a set A

$\mathrm{ri}(A)$	relative interior of a set A
$J(f)(x)$	Jacobian matrix of a function at x
X^*	dual cone of the cone X
$\Lambda(f,\alpha)$	lower level set of a function f at level α
$\mathbb{N}$	set of all natural numbers
$N_C(x)$	normal cone to a set C at x
0^+X	recession cone of a set X
$\mathbb{R}$	set of all real numbers
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty, -\infty\}$
$\mathbb{R}_+$	set of all nonnegative real numbers
$\sigma_C(.)$	support function of a set C
$\mathbb{R}_{++}$	set of all positive real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^n_+$	$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \; \forall \; i\}$
$\mathbb{R}^n_{++}$	$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > 0 \; \forall \; i\}$
$\Omega(f,\alpha)$	upper level set of a function f at level α

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