COLLEGE ALGEBRA

WENTWORTH

A
COLLEGE ALGEBRA

BY

G. A. WENTWORTH

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

REVISED EDITION

GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

PRINTED IN THE UNITED STATES OF AMERICA

526.4

The Athenæum Press
GINN AND COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE

This work, as the name implies, is intended for colleges and scientific schools. The first part is simply a review of the principles of Algebra preceding Quadratic Equations, with just enough examples to illustrate and enforce these principles. By this brief treatment of the first chapters sufficient space is allowed, without making the book cumbersome, for a full discussion of Quadratic Equations, The Binomial Theorem, Choice, Chance, Series, Determinants, and The General Properties of Equations. Every effort has been made to present in the clearest light each subject discussed, and to give in matter and methods the best training in algebraic analysis at present attainable. Many problems and sections can be omitted at the discretion of the instructor.

The author is under great obligation to J. C. Glashan, LL.D., Ottawa, Canada, to Professor J. J. Hardy, Ph.D., Lafayette College, Easton, Pa., and to W. H. Butts, A.M., Michigan University, Ann Arbor, Mich., who have read the proofs and given valuable suggestions on the subject-matter.

Answers to the problems are bound separately in paper covers, and will be furnished free to pupils when *teachers* apply to the *publishers* for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

Exeter, N.H., May, 1902.

TABLE OF CONTENTS

COLLEGE ALGEBRA

CHAPTER I

FUNDAMENTAL IDEAS

1. Magnitude, Quantity, and Number. Whatever admits of increase or decrease is called a **magnitude**. Every magnitude must therefore admit of comparison with another magnitude of the same kind in such a way as to determine whether the first is greater than, less than, or equal to the other.

A measurable magnitude is a magnitude that admits of being considered as made up of parts all equal to one another.

To **measure** any given measurable magnitude, we take as standard of reference a definite magnitude of the same kind as the magnitude to be measured and determine how many magnitudes, each equal to the standard of reference, will together constitute the given magnitude.

A **quantity** is a measurable magnitude expressed as a *magnitude actually measured.* Hence, the expression of a quantity consists of two components. One of these components is the *name* of the magnitude that has been selected as the standard of reference or measurement. The other component expresses how many magnitudes, each equal to the standard of reference, must be taken to make up the quantity. The standard magnitude is termed a **unit**, and the other component of the expression is termed the **numerical value** of the expression. Hence,

A **unit** is the standard magnitude employed in counting any collection of objects or in measuring any magnitude.

A **number** is that which is applied to a unit to express how many parts, each equal to the unit, there are in the magnitude measured.

The endless succession of numbers *one, two, three, four,* etc., employed in counting is called the **natural series of numbers.**

2. In the statement *James walked 12 miles,* the number of miles is actually stated, and the 12 is therefore called a **known number**, or it is said to be *explicitly assigned.*

In the statement *If from five times the number of miles James walked, ten is subtracted, the remainder will be fifty,* the number of miles, though not *directly* given, may be found from the *data* to be twelve and is therefore said to be implied in the statement, or it is called an *implicitly assigned* number, or more commonly, an **unknown number.**

In the statement *If from the double of a number six is subtracted, the result will be the same as if three had been subtracted from that number and the remainder doubled,* the number to be doubled is assigned neither *explicitly* nor *implicitly,* since the statement is true for any number whatever. A number of this kind, which may have any value whatever, is called an *arbitrary number.* Arbitrary numbers are frequently called **known numbers**, as they are often assumed to be known, though not definitely assigned.

3. Numbers explicitly assigned are represented in Algebra, as they are in Arithmetic, by the numerals or figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and combinations of these. Each figure or combination of figures represents one and but one particular number. Numbers implicitly assigned and arbitrary numbers are usually represented by the letters of the alphabet. The first letters of the alphabet, as a, b, c, etc., are generally used to represent arbitrary numbers, while z, y, x, w, etc., commonly represent unknown numbers.

4. When any letter, as x, is used in the course of a calculation it denotes the same number throughout. We may also represent different numbers by the same letter with marks affixed.

Thus, instead of writing a, b, c for three different numbers, we may represent these numbers by the symbols a_1, a_2, a_3 (read *a sub-one*, *a sub-two*, etc.), or by a', a'', a''' (read *a prime*, *a second*, etc.).

5. In Arithmetic the figures that represent numbers are generally themselves called numbers; and, similarly, in Algebra the symbols that stand for numbers are themselves called numbers. Letter-symbols are called *literal expressions*, and figure-symbols *numerical expressions*.

The number which a letter represents is called its *value*, and if represented *arithmetically*, its *numerical* value.

6. In elementary Algebra we consider all quantities as expressed numerically in terms of some unit, and the symbols represent only the *purely numerical parts* of such quantities. In other words, the symbols denote what are called in Arithmetic *abstract* numbers.

7. An **algebraic expression** is the expression of a number in algebraic symbols.

8. Certain words and phrases occur so often in Algebra that it is found convenient to represent them by easily made symbols.

Symbols of Relation.

$=$, read *equals, is equal to, will be equal to*, etc.

$\neq$, read *is not equal to*, etc.

$>$, read *is greater than*, thus $9 > 4$.

$<$, read *is less than*, thus $4 < 9$.

$:$, $::$, the signs of proportion, as in Arithmetic.

Thus, $a : b :: c : d$, or $a : b = c : d$, is read a *is to* b *as* c *is to* d.

Symbols for Words.

$\therefore$, read *therefore, consequently, hence.*

$\because$, read *because, since.*

Thus, $\because a = b$, and $b = c$; $\therefore a = c$, is read *since* a *equals* b, *and* b *equals* c; *therefore* a *equals* c.

$\cdots$, the symbol of continuation, is read *continued by the same law.*

Thus, 1, 2, 3, 4, $\cdots$ means that we are to continue the numbers by the same law; x_1, x_2, x_3, $\cdots$, x_n means x_1, x_2, x_3, x_4, x_5, and so on to x_n.

9. Signs of Operation. The principal operations of Algebra are Addition, Subtraction, Multiplication, Division, Involution, Evolution, and Logarithmation. A mark used to denote that one of these operations is to be performed on a number is called a *sign of operation.* These *signs of operation* will now be explained.

10. The **sign of addition** is $+$ (read *plus*). As in Arithmetic, it denotes that the number before which it stands is an *addend.*

Thus, $a + b$ means that b is to be added to a; so that if a represents 6 and b represents 4, $a + b$ represents $6 + 4$, which is 10. $a + b + c$ denotes that b is to be added to a, and then c added to their sum.

The sum of two or more numbers is expressed by writing them in a row with the sign $+$ *before each of them except the first number.*

11. The **sign of subtraction** is $-$ (read *minus*). As in Arithmetic, placed before a number it denotes that that number is a *subtrahend.*

Thus, $a - b$ (read a *minus* b), indicates that the number represented by b is to be subtracted from the number represented by a; so that, if a represents 6 and b represents 4, $a - b$ is equivalent to $6 - 4$, which is 2.

Hence, to indicate that a number is to be subtracted from another number, as a from x, *write the subtrahend after the minuend with the sign* $-$ *between them.*

The expression $a + b - c$ denotes that b is to be added to a, and then c subtracted from the sum. $a - b - c$ denotes that b is to be subtracted from a, and then c subtracted from the remainder.

12. Numbers to be multiplied together are called **factors**, and the resulting number is called the *product* of these factors. Multiplication is indicated in two ways:

1. *By a sign of operation.* 2. *By position.*

The **signs of multiplication** are $\times$ and $\cdot$ (read *into, times,* or *multiplied by*).

Thus, $3 \cdot 4 \cdot 5$, or $3 \times 4 \times 5$ indicates the *continued* product of the three factors 3, 4, and 5. In like manner, $a \cdot b$, or $a \times b$, indicates the product of the factors a and b.

If all the factors or all but one are represented by letters, the *signs of operation*, $\times$ and $\cdot$, are generally omitted; this method is called **indicating multiplication by position.**

Thus, five times a is written $5a$ (read *five a*), and $\frac{3}{8}$ of the product of m and z is written $\frac{3}{8}mz$.

A number which multiplies another number is called a **coefficient** of that number. A *coefficient* (literally, *co-factor*) is therefore simply a multiplier, *numerical* or *literal*.

Thus, in the expression $5amx$,

5	is the numerical coefficient of			amx,
$5a$	"	literal	"	" mx,
$5am$	"	"	"	" x.

If no numerical coefficient is written, unity is understood as the actual numerical coefficient.

13. The **sign of division** is $\div$ (read *divided by*), and denotes that the number immediately following it is a divisor.

Thus, $a \div b$ (read *a divided by b*) means that a is to be divided by b. If a represents 12 and b represents 4, $a \div b$ represents $12 \div 4$, or 3.

Division is also indicated by arranging the numbers in the form of a fraction with the *dividend* for *numerator* and the *divisor* for *denominator.*

Thus, $a \div b$ may be written $\frac{a}{b}$; $ax \div by$ may be written $\frac{ax}{by}$.

This method is called **indicating division by position.**

14. In an expression such as $7ax + 5cy - 3dz$ (read *seven* ax *plus five* cy *minus three* dz) *the multiplications are to be performed before the additions and subtractions.*

In an expression such as $\frac{ax}{m} + \frac{by}{n} - \frac{cz}{q}$ *the multiplications and divisions are to be performed before the additions and subtractions,* so that in this expression the quotient of ax by m is to be increased by the quotient of by by n, and the sum diminished by the quotient of cz by q.

15. **A power** of a number is the product obtained by using that number a certain number of times as a multiplier, starting with *unity as first multiplicand.* The operation of forming a power is called **involution**; the number used as a multiplier is called the **base** of the power; the *number of successive multiplications by the base* is called the **degree** of the power; and the number indicating the degree of the power is called the **exponent** or **index** of the power and is written in small characters to the right and a little above the line of the base.

Thus, $1 \times a \times a$ is represented by a^2 (read *a square*); here a is the *base*, 2 is the *exponent* (or *index*), and a^2 is the second *power* of a.

$1 \cdot c \cdot c \cdot c$ is represented by c^3 (read *c cube*); here c is the base, 3 is the exponent, and the number c^3 is the third power of c.

In x^5 (read *x to the fifth*), x is the base, 5 is the exponent, and the number x^5 is the fifth power of x.

Since the exponent denotes how many multiplications by the base are to be made, the first to be performed on unity, it follows that a^1, the first power of a, represents $1 \times a$, or simply a.

Hence, also, a^0, the zero power of a, denotes that *no* multiplication by a is to be made, or, in other words, that the unit-multiplicand is not to be multiplied by a. Therefore $a^0 = 1$ for any value of a whatsoever.

16. In writing a power at full length as a product it is usual to omit the unit-multiplicand, just as it is usual to omit a unit-coefficient where such occurs.

Thus, instead of writing $x^3 = 1 \times x \times x \times x$, we write $x^3 = x \times x \times x$.

In this method of expressing the value of a power *the exponent denotes the number of times the base is taken as a factor.*

17. Comparing powers, the second power is said to be *higher* than the first, the third higher than the second, etc.

18. In an expression such as $4\,a^2b^3 \div c^2$ (read 4 a *square* b *cube divided by* c *square*) *the involutions are to be performed before the multiplications and divisions.*

19. Involution is the operation of forming a power by taking the same number several times as a factor.

Evolution is the inverse of Involution, or the operation of finding one of the *equal factors* of a number. A **root** is one of the equal factors. If the number is resolved into *two equal factors,* each factor is called the **square root**; if into three equal factors, each factor is called the **cube root**; if into four equal factors, each factor is called the **fourth root**; and so on.

The root sign is $\sqrt{}$. Except for the square root, a number-symbol is written over the root sign to show into how many equal factors the given number is to be resolved. This number-symbol is called the **index of the root.**

Thus, $\sqrt{64}$ means the square root of 64; $\sqrt[3]{64}$ means the cube root of 64.

20. Logarithmation is the operation of determining the index or exponent which the given base must have in order that the resulting root or power may be equal to a given number. The index or exponent is called the **logarithm** of the given number to the given base.

Thus, if a and b are given numbers and $a^n = b$, n is called the logarithm of b to the base a.

21. Positive and Negative Numbers. There are quantities which stand to each other in such an opposite relation that, when combined, they cancel each other entirely or in part.

Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and leave four dollars gain.

An opposition of this kind exists in *assets* and *debts,* in motion *forwards* and motion *backwards,* in motion *to the right* and motion *to the left,* in the rise *above* zero and the fall *below* zero of the mercury of a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards* along the natural series of numbers. If we wish to subtract 4 from 6, we start at 6 in the natural series, count four units backwards, and arrive at two, the difference sought. If we subtract 6 from 6, we start at 6 in the natural series, count six units backwards, and arrive at zero. If we try to subtract nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers has come to an end.*

22. In order to subtract a greater number from a smaller, it is necessary to *assume* a new series of numbers, beginning at zero and extending backwards. If the natural series advances from zero to the right, by repetitions of the unit, the new series must recede from zero to the left, by repetitions of the unit; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign +; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign −. The two series of numbers will be written thus:

$$\cdots -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4, \cdots$$

and may be considered as forming but a single series consisting of a positive portion or branch, a negative portion or branch, and zero. The complete series thus formed is called the **scalar series.**

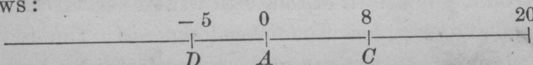
If, now, we wish to subtract 9 from 6, we begin at 6 in the positive branch, count nine units in the *negative direction* (to the left), and arrive at -3 in the negative branch. That is, $6 - 9 = -3$.

The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

If a and b represent any two numbers of the positive branch, the expression $a - b$ will denote a positive number when a is greater than b; will be equal to zero when a is equal to b; will denote a negative number when a is less than b.

If we wish to add 9 to -6, we begin at -6 in the negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$ in the positive branch.

We may illustrate the use of positive and negative numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? Answer: at C, a point 8 feet to the right of A. That is, 20 feet – 12 feet = 8 feet; or, $20 - 12 = 8$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D, a point 5 feet to the left of A. That is, if we consider distance measured in feet to the left of A as forming a negative series of numbers, beginning at A, $20 - 25 = -5$. Hence, the phrase, 5 feet to the left of A, is now expressed by the negative number -5.

23. Numbers with the sign + or – are called **scalar numbers.** They are unknown in elementary Arithmetic, but play a very important part in Algebra. Numbers regarded without reference to the signs + or – are called **absolute numbers.**

Every algebraic number, as $+4$ or -4, consists of a sign + or – and the absolute value of the number; in this case 4. The sign shows whether the number belongs to the positive or the negative series of numbers; the absolute value shows what place the number has in the positive or the negative series.

When no sign stands before a number the sign + is always understood.

Thus, 4 means the same as $+4$, a means the same as $+a$.

But *the sign — is never omitted.*

Two numbers which have, one the sign + and the other the sign —, are said to have **unlike signs.**

Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined.

Thus, $+4-4=0$, $+a-a=0$.

24. Meaning of the Signs. The use of the signs + and —, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense they are signs of *operations* and are common to both Arithmetic and Algebra. In the second sense they are signs of *opposition* and are employed in Algebra alone.

25. When an expression is made up of several parts connected by the signs +, —, each of these parts taken with the sign immediately preceding it (+ being understood if no written sign precedes) is called a **term.**

Thus, $a+b-c+d+e$ consists of the five *terms* $+a$, $+b$, $-c$, $+d$, $+e$.

A term whose sign is + is called a **positive term**; a term whose sign is — is called a **negative term.**

An expression which consists of but one term is called a **monomial** or **simple expression.**

An expression which consists of two or more terms is called a **polynomial** or **compound expression.**

A polynomial of two terms is called a *binomial.* A polynomial of three terms is called a *trinomial.* Polynomials of three or more terms are sometimes called *multinomials.*