Comparison Theorems and Submanifolds in Finsler Geometry

（芬斯勒几何中的比较定理与子流形）

Bingye Wu（吴炳烨）

SCIENCE PRESS
Beijing

Comparison Theorems and Submanifolds in Finsler Geometry

Bingye Wu (吴炳烨)

Science Press
Beijing

Responsible Editor: jingke Li

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-043436-4

Preface

Finsler geometry is just the Riemannian geometry without the restriction of quadratic form, and it was originated in Riemann's 1854 "habilitation" address: "Uber die Hypothesen, welche der Geometrie zu Grunde liegen" (On the Hypotheses, which lie at the Foundation of Geometry). In the context of Riemann's lecture, the restriction to a quadratic form constitutes only a special case. Riemann saw the difference between the quadratic case and the general case, and chose the former case as the representative to understand the structure of manifolds. He saw the great merit of the former case, and introduced for it the curvature tensor and the notion of sectional curvature. Such was done through a Taylor expansion of the Riemannian metric. There was no significant progress in the general case until 1918, when Paul Finsler studied the variation of general case. For this reason, the general case is referred to as Finsler metric. Since the end of 20th century, Finsler geometry has made great progress under the leadership of the late great geometer Shiing Shen Chern, and it has gotten many applications in various fields such as Control Theory and Relativity Theory, etc.

In this monograph we focus our interest on comparison theorems and submanifolds in Finsler geometry, and systematically describe the author's contributions in these topics. We express the discussion in the thread of volume form. Volume, as an important geometric invariant, plays a key role in global differential geometry, and it is closely related to the curvature and topology of differential manifolds. It should be pointed out here that volume form is uniquely determined by the given Riemannian metric, while there are different choices of volume forms for Finsler metrics. Thus it is important to choose suitable volume forms in the research of Finsler geometry. The frequently used volume forms in the literature are so-called Busemann-Hausdorff volume form and Holmes-Thompson volume form. We shall introduce the extreme volume form, including the maximal and minimal volume forms, for Finsler metric. By using the extreme volume form, we are able to remove the addition assumption on S-curvature that is needed in the literatures. This shows that the extreme volume form is a good choice in comparison technique in Finsler geometry. As for the theory of Finsler submanifold, Professor Zhongmin Shen first introduced the notions of the mean curvature and minimal submanifolds for Finsler submanifolds by using the volume variation with respect to Busemann-Hausdorff volume form; here we shall generalize this theory to general Finsler volume form.

The monograph consists of four chapters. In Chapter 1 we give a brief description of basic concepts and fundamental results in Finsler geometry, including the Minkowski spaces, geodesics, Jacobi fields, conjugate points and the basic index lemma, which are necessary to study global Finsler geometry. Normally the computations in Finsler geometry are very complicated; here we discuss without using the structure equations so that the computations are relatively simple and the reader can easily understand the arguments. It is also worth pointing out that we provide a new and simple proof for Deicke's theorem: The Minkowski space is Euclidean if and only if its mean Cartan tensor vanishes. Then in Chapter 2 we focus on comparison theorems in Finsler geometry. Firstly, we derive the basic comparison theorem—the Rauch's comparison theorem. Secondly, we introduce the notions of general Finsler volume form, S-curvature, Hessian and Laplacian of smooth functions, and then establish the Hessian comparison theorems, Laplacian comparison theorems and volume comparison theorems for Finsler manifolds under various curvature conditions. We also establish a toponogov type comparison theorem for geodesic triangles. It should be noted that the notion of Hessian defined here is different from that defined by Professor Zhongmin Shen. The advantage of our definition is that the Hessian is a symmetric bilinear form and we can treat it by using the theory of symmetric matrices. Chapter 3 is comprised of various applications of comparison theorems, mainly on the curvature and topology of Finsler manifolds. We first derive a generalized Myers theorem for Finsler manifold, and generalize Calabi-Yau's linear volume growth theorem and Mckean type estimations of the first eigenvalue to Finsler manifolds. Then we discuss the Gromov's precompactness theorem, the first Betti number, and the fundamental group of Finsler manifolds under suitable curvature assumptions. We also obtain a lower bound for injectivity radius for compact reversible Finsler manifolds, and discuss a fundamental property of finite topological type. It should be stressed here that the counting function is a basic notion to describe the growth of fundamental group, and the usual definition of counting function used in the literature demands that the fundamental group be finitely generated. Here we provide a new definition which removes the above restriction. As a consequence, many results are new even for Riemannian manifolds. Finally in Chapter 4 we establish the theory of submanifolds for general volume form and obtain some basic results. This monography may be served as a textbook or a reference book for postgraduate students and scientists who are interested in global Finsler geometry.

I would like to take this opportunity to thank my advisors Professor Yi-Bing Shen and Professor Yuan-Long Xin for their constant supports and encouragements. I would also like to thank several people in my research experience in Finsler geometry. They are: Professor Zhongmin Shen, Professor Xiaohuan Mo and Professor

Xinyue Cheng. Finally, I thank my wife Wei-Cheng Yu, for her constant support and understanding. The project is supported by the Natural Science Foundation of China (Grant No. 11171139).

Bingye Wu
Jan. 2015

Contents

Chapter 1

Basics on Finsler Geometry

Finsler geometry is just the Riemannian geometry without the quadratic restriction. Instead of an Euclidean norm on each tangent space, one endows every tangent space of a differentiable manifold with a Minkowski norm. Since there are many different Minkowski spaces that are not mutually isomorphic, Finsler manifolds are more "colorful" than Riemannian manifolds. In this chapter we shall give a brief description of basic quantities and fundamental properties for Finsler metrics which are foundations to study global Finsler geometry.

1.1 Minkowski Space

1.1.1 Definition and Examples

Before giving the definition of Minkowski space, let us first discuss a useful property of positively homogeneous function on n-dimensional space of real number $\mathbb{R}^n$. Recall that a function $f:\mathbb{R}^n\to\mathbb{R}$ is called a *positively homogeneous function of degree* s on $\mathbb{R}^n$ if $f(\lambda y)=\lambda^s f(y)$ holds for any $y\in\mathbb{R}^n$ and $\lambda>0$.

Lemma 1.1(Euler's Lemma) *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a positively homogeneous function of degree s on $\mathbb{R}^n$, then $f_{y^i}(y)y^i=sf(y)$, here we have used the the Einstein convention, that is, repeated indices with one upper index and one lower index denotes summation over their ranges, and f_{y^i} denotes the partial derivative of f with respect to y^i.*

Proof Taking derivative on two sides of $f(\lambda y)=\lambda^s f(y)$ with respect to λ yields $f_{y^i}(\lambda y)y^i=s\lambda^{s-1}f(y)$. Letting $\lambda=1$ we get the desired result. □

Now we give the definition of Minkowski space.

Definition 1.1 Let V be an n-dimensional real vector space. A function $F=F(y)$ on V is called a *Minkowski norm* if it satisfies the following properties:

(1) $F(y)\geqslant 0$ for any $y\in V$, and $F(y)=0$ if and only if $y=0$;

(2) $F(\lambda y)=\lambda F(y)$ for any $y\in V$ and $\lambda>0$;

(3) F is C^∞ on $V\backslash\{0\}$ such that for any $y\in V\backslash\{0\}$, the following bilinear symmetric functional $\mathbf{g}_y$ on V is an inner product:

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2(y+su+tv)\right]_{s=t=0}.$$

The pair (V,F) is called a *Minkowski space*, and $\mathbf{g}_y$ the *fundamental form with respect to* y . If $F(y)=F(-y)$ holds for any $y\in V$, then (V,F) is called *reversible.*

Let (V,F) be a Minkowski space, and

$$S=\{y\in V|F(y)=1\}.$$

S is a closed hypersurface around the origin, which is diffeomorphic to the standard sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$. S is called the *indicatrix* of (V,F). Fix a basis $\{\mathbf{b}_i\}$ of V, and view $F(y)=F(y^i\mathbf{b}_i)$ as a function of $(y^i)\in\mathbb{R}^n$. For $y\neq 0$, write

$$g_{ij}(y) := \mathbf{g}_y(\mathbf{b}_i,\mathbf{b}_j)=\frac{1}{2}\left[F^2\right]_{y^iy^j}(y)=F_{y^i}(y)F_{y^j}(y)+F(y)F_{y^iy^j}(y). \tag{1.1}$$

Then

$$\mathbf{g}_y(u,v)=g_{ij}(y)u^iv^j,\quad u=u^i\mathbf{b}_i, v=v^i\mathbf{b}_i. \tag{1.2}$$

Since the Minkowski norm is a positively homogeneous function of degree one, by Euler's lemma we get

$$F(y)=\sqrt{g_{ij}(y)y^iy^j},\quad y=y^i\mathbf{b}_i.$$

When (V,F) is reversible, it is clear that $g_{ij}(y)=g_{ij}(-y)$, and consequently $\mathbf{g}_y(u,v)$ $=\mathbf{g}_{-y}(u,v)$.

For any $y\neq 0$, one has the following decomposition for V:

$$V=\mathbb{R}\cdot y\oplus W_y,$$

here W_y is the orthogonal complementary of $\mathbb{R}\cdot y$ in V with respect to the inner product $\mathbf{g}_y$, namely,

$$W_y=\{w\in V|\mathbf{g}_y(y,w)=0\}\subset V.$$

Let

$$\mathbf{h}_y(u,v) := \mathbf{g}_y(u,v)-\frac{1}{F^2(y)}\mathbf{g}_y(y,u)\mathbf{g}_y(y,v).$$

We call $\mathbf{h}_y$ *the angular form with respect to* y, and its components are

$$h_{ij}(y) := \mathbf{h}_y(\mathbf{b}_i,\mathbf{b}_j)=g_{ij}-\frac{1}{F^2}g_{ip}y^pg_{jq}y^q=FF_{y^iy^j}.$$

It is easy to know that $\mathbf{h}_y(y,u)=0,\forall u\in V$, and for $u=w+\lambda y\in V$ with $w\in W_y$, one has

$$\mathbf{h}_y(u,u)=\mathbf{g}_y(w,w)\geqslant 0,$$

and the equality holds if and only if $u = \lambda y$. Hence, $\mathbf{h}_y$ is semi-positive definite on V while it is positive definite on W_y. Now we prove two fundamental inequalities for Minkowski space.

Lemma 1.2 *Let (V, F) be a Minkowski space, then*

(1) *F satisfies*

$$F(u+v) \leqslant F(u) + F(v), \quad u, v \in V, \tag{1.3}$$

and the equality holds if and only if $u = 0$ or there is $\lambda \geqslant 0$ such that $v = \lambda u$;

(2) (Cauchy-Schwarz Inequality) *Let $y \neq 0$, then for any $u \in V$ one has*

$$\mathbf{g}_y(y, u) \leqslant F(y)F(u), \tag{1.4}$$

and the equality holds if and only if there exists $\lambda \geqslant 0$ such that $u = \lambda y$.

Proof Let us first prove (1.3). Without loss of generality, we can assume that $u, v \in V\backslash\{0\}$. If u, v are linearly independent, let $y(t) = tu + (1-t)v$, then $y(t) \neq 0, t \in [0,1]$. Consider the function $\varphi(t) = F(y(t))$, then

$$\varphi''(t) = F_{y^i y^j}(y(t))(u^i - v^i)(u^j - v^j) = \frac{1}{F(y(t))}\mathbf{h}_{y(t)}(u-v, u-v) > 0.$$

Therefore, $\varphi = \varphi(t)$ is a strict convex function, and we see from the property of convex function that

$$2\varphi\left(\frac{1}{2}\right) < \varphi(0) + \varphi(1),$$

namely,

$$F(u+v) < F(u) + F(v).$$

When u, v are linearly dependent, then there exists $\lambda \in \mathbb{R}$ such that $v = \lambda u$. If $1 + \lambda \geqslant 0$, then

$$F(u+v) = F((1+\lambda)u) = (1+\lambda)F(u) \leqslant F(u) + F(\lambda u) = F(u) + F(v),$$

and the equality holds if and only if $\lambda \geqslant 0$. Finally, when $1 + \lambda < 0$,

$$F(u+v) = F(-(1+\lambda)(-u)) = -(1+\lambda)F(-u) = -F(-u) + F(\lambda u) < F(u) + F(v),$$

and thus (1.3) is proved.

Now let us prove (1.4). For $w \in W_y$, let $\varphi(t) = F^2(y + tw)$, then by (1.1),(1.2) and Euler's lemma we have

$$\varphi'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\left[F^2(y+tw)\right]|_{t=0} = 2F(y)F_{y^i}(y)w^i = 2\mathbf{g}_y(y, w) = 0,$$

$$\varphi''(t) = 2\left[F_{y^i}(y+tw)w^i\right]^2 + 2F(y+tw)F_{y^i y^j}(y+tw)w^i w^j \geqslant 2\mathbf{g}_{y+tw}(w, w) \geqslant 0,$$

and the equality holds if and only if $w = 0$. Consequently, $\varphi(0) \leqslant \varphi(t), \forall t \neq 0$, and the equality holds if and only if $w = 0$. Especially,

$$F(y) \leqslant F(y + w), \quad w \in W_y, \tag{1.5}$$

with the equality holds if and only if $w = 0$. Now for any $u \in V$, write $u = \lambda y + w$, here $\lambda \in \mathbb{R}, w \in W_y$. Then

$$\mathbf{g}_y(y, u) = \lambda \mathbf{g}_y(y, y) = \lambda F^2(y). \tag{1.6}$$

If $\lambda \leqslant 0$, then it is easy to see from (1.6) that (1.4) holds, with the equality holds if and only if $\lambda = 0$ and $u = 0$. On the other hand, if $\lambda > 0$, it is clear from (1.5) and (1.6) that

$$\mathbf{g}_y(y, u) = \lambda F(y) F(y) \leqslant \lambda F\left(y + \frac{1}{\lambda} w\right) F(y) = F(u) F(y).$$

The equality holds if and only if $w = 0$, namely, $u = \lambda y, \lambda \geqslant 0$, so the lemma is proved. □

From (1.3) it is clear that F is the norm of vector space V in the usual sense when F is reversible, and it is also easy to verify that (1.4) can be rewritten as

$$|\mathbf{g}_y(y, u)| \leqslant F(y) F(u) \tag{1.4$'$}$$

when (V, F) is reversible. But (1.4)$'$ does not hold for general Minkowski space (see Example 1.2).

In the following we give some important examples of Minkowski space.

Example 1.1(Euclidean norm) Let $\langle , \rangle$ is an inner product on vector n-space V, and $\{\mathbf{b}_i\}$ is a basis of V. Let

$$\alpha = \sqrt{\langle y, y \rangle} = \sqrt{a_{ij} y^i y^j}, \quad y = y^i \mathbf{b}_i,$$

where $a_{ij} = \langle \mathbf{b}_i, \mathbf{b}_j \rangle$. It is obvious that α is a Minkowski norm on V, and $\mathbf{g}_y(u, v) = \langle u, v \rangle$ is independence of $y \in V$. α is called the *Euclidean norm*, and (V, α) is called the *Euclidean space*. Euclidean norm is reversible, and it is well known that all Euclidean spaces with the same dimension are mutually isomorphic isometrically. The standard Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ is

$$|y| = \sqrt{\sum_{i=1}^{n} \left(y^i\right)^2}, \quad y = (y^i) \in \mathbb{R}^n.$$

Example 1.2(Randers norm) Let $\alpha = \sqrt{\langle y, y \rangle} = \sqrt{a_{ij} y^i y^j}$ be an Euclidean norm on vector space V, and $\beta = b_i y^i \in V^*$ a linear functional on V. Let

$$F(y) = \alpha(y) + \beta(y). \tag{1.7}$$

A direct computation shows that

$$g_{ij} = \frac{F}{\alpha}\left\{a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha} + \frac{\alpha}{F}\left(b_i + \frac{y_i}{\alpha}\right)\left(b_j + \frac{y_j}{\alpha}\right)\right\}, \tag{1.8}$$

$$\det(g_{ij}) = \left(\frac{\alpha+\beta}{\alpha}\right)^{n+1}\det(a_{ij}), \tag{1.9}$$

where $y_i = a_{ik}y^k$. Let

$$\|\beta\|_\alpha := \sup_{y\in V\setminus\{0\}} \frac{\beta(y)}{\alpha(y)} = \sup_{\alpha(y)=1} \beta(y),$$

then it is easy to know from the method of Lagrange multipliers that $\|\beta\|_\alpha = \sqrt{a^{ij}b_ib_j}$, here $(a^{ij}) = (a_{ij})^{-1}$. By (1.9) we see that the matrix (g_{ij}) is positive if and only if $\|\beta\|_\alpha < 1$. Therefore, F is a Minkowski norm on V if and only if $\|\beta\|_\alpha < 1$. In this situation, we call F a *Randers norm* on V. Randers norm is non-reversible when $\beta \neq 0$.

Now we consider the Randers norm on $\mathbb{R}^n$ given by $F(y) = |y| + by^1, 0 < b < 1$. Particularly, choose $y = (1, 0, \cdots, 0) = -u$, then $F(y) = 1 + b, F(u) = 1 - b$, and from (1.8) one see that $\mathbf{g}_y(y,u) = -(1+b)^2$. Consequently $|\mathbf{g}_y(y,u)| = (1+b)^2 > (1+b)(1-b) = F(y)F(u)$, that is to say, (1.4)′ does not hold.

Example 1.3 Consider the function on the 2-plane $\mathbb{R}^2$ as following:

$$F(y) = \left((y^1)^4 + 3c(y^1)^2(y^2)^2 + (y^2)^4\right)^{\frac{1}{4}}, \quad y = (y^1, y^2) \in \mathbb{R}^2.$$

A direct calculation shows that

$$g_{11} := \frac{1}{2}\left(F^2\right)_{y^1y^1} = \frac{2(y^1)^6 + 9c(y^1)^4(y^2)^2 + 6(y^1)^2(y^2)^4 + 3c(y^2)^6}{2F^6},$$

$$g_{12} := \frac{1}{2}\left(F^2\right)_{y^1y^2} = \frac{(9c^2-4)(y^1)^3(y^2)^3}{2F^6} = g_{21},$$

$$g_{22} := \frac{1}{2}\left(F^2\right)_{y^2y^2} = \frac{3c(y^1)^6 + 6(y^1)^4(y^2)^2 + 9c(y^1)^2(y^2)^4 + 2(y^2)^6}{2F^6},$$

$$\det(g_{ij}) = g_{11}g_{22} - (g_{12})^2 = \frac{3\left(2c(y^1)^4 + (4-3c^2)(y^1)^2(y^2)^2 + 2c(y^2)^4\right)}{4F^4}.$$

It is clear that $g_{11} > 0$ and $g_{22} > 0$ holds if and only if $c > 0$. Now assume that $c > 0$, and note that

$$2c(y^1)^4 + (4-3c^2)(y^1)^2(y^2)^2 + 2c(y^2)^4 = 2c\left((y^1)^4 + 2\delta(y^1)^2(y^2)^2 + (y^2)^4\right),$$

where $\delta := (4-3c^2)/(4c)$. Thus $\det(g_{ij})$ is positive for any $y \neq 0$ if and only if $\delta > -1$, and $\delta > -1$ is equivalent to $c < 2$. In summary, F is a Minkowski norm on $\mathbb{R}^2$ if and only if $0 < c < 2$, and clearly it is reversible.

1.1.2 Legendre Transformation

Let (V, F) be a Minkowski space, and $\mathbf{b}_1, \cdots, \mathbf{b}_n$ a basis of V. Let V^* be the dual vector space of V with the dual basis $\theta^1, \cdots, \theta^n$. The *Legendre transformation* $l : V \to V^*$ from V to V^* is

$$l(y) := \begin{cases} \mathbf{g}_y(y, \cdot) \in V^*, & \forall y \in V\backslash\{0\}, \\ 0, & y = 0. \end{cases}$$

It is easy to see from the definition that the expression of Legendre transformation is given by

$$l(y) = g_{ij}(y) y^j \theta^i, \quad y = y^i \mathbf{b}_i \in V\backslash\{0\}. \tag{1.10}$$

For any $\xi \in V^*$, define

$$F^*(\xi) := \sup_{y \in V\backslash\{0\}} \frac{\xi(y)}{F(y)} = \sup_{y \in S} \xi(y), \tag{1.11}$$

then it is obvious that

$$F^*(\lambda\xi) = \lambda F^*(\xi), \quad F^*(\xi + \eta) \leqslant F^*(\xi) + F^*(\eta), \quad \xi, \eta \in V^*, \lambda > 0.$$

We call F^* the *dual norm* of F. We have following lemmas for Legendre transformation.

Lemma 1.3 *The Legendre transformation $l : V\backslash\{0\} \to V^*\backslash\{0\}$ is a smooth diffeomorphism from $V\backslash\{0\}$ to $V^*\backslash\{0\}$, and for any $y \in V\backslash\{0\}$, the covector $\xi = l(y) = \mathbf{g}_y(y, \cdot) \in V^*$ satisfies*

$$F(y) = F^*(\xi) = \frac{\xi(y)}{F(y)}. \tag{1.12}$$

Proof From the expression (1.10) we see that the Legendre transformation l is a smooth map from $V\backslash\{0\}$ to $V^*\backslash\{0\}$. In order to prove that it is a diffeomorphism, we need to verify that it is both injective and surjective. Suppose that there exist $y, u \in V\backslash\{0\}$ such that $l(y) = l(u)$, i.e.,

$$\mathbf{g}_y(y, w) = \mathbf{g}_u(u, w), \quad \forall w \in V.$$

Set $w = u$ we obtain

$$F^2(u) = \mathbf{g}_u(u, u) = \mathbf{g}_y(y, u) \leqslant F(y)F(u),$$

and consequently,

$$F(u) \leqslant F(y).$$

Similarly,

$$F(y) \leqslant F(u).$$

Therefore, $F(y) = F(u)$, and

$$\mathbf{g}_y(y, u) = F^2(u) = F(y)F(u),$$

which together with Lemma 1.2 yields $y = u$, and hence l is injective. Now we prove that l is also surjective. Note that $S = \{y \in V | F(y) = 1\}$ is a closed hypersurface of V, for any $\xi \in V^* \backslash \{0\}$, there exists $u \in S$ such that

$$\xi(u) = F^*(\xi) = \sup_{y \in S} \xi(y).$$

Let $y = \xi(u)u$, then

$$F(y) = \xi(u) = F^*(\xi) = \frac{\xi(y)}{F(y)}. \tag{1.13}$$

If $\xi = l(y)$, then l is surjective, and the last formula is just (1.12). In the following we want to prove $\xi = l(y)$, i.e.,

$$\xi(v) = \mathbf{g}_y(y, v), \quad \forall v \in V.$$

For any $t \in \mathbb{R}$, if $y + tv \neq 0$, then

$$\xi\left(\frac{y + tv}{F(y + tv)}\right) \leqslant F^*(\xi) = \xi(u),$$

and consequently,

$$\xi(y + tv) \leqslant F(y + tv)\xi(u).$$

On the other hand, above inequality obviously holds when $y + tv = 0$, and from (1.13) we see that above inequality becomes an equality when $t = 0$. As the result, $f = f(t) := \xi(y + tv) - F(y + tv)\xi(u) \leqslant 0$, and f attains its maximum at $t = 0$. From (1.1) and Euler's lemma we have

$$\mathbf{g}_y(y, v) = g_{ij}(y)y^i v^j = F(y)F_{y^i}(y)v^i,$$

which together with the maximum principle yields

$$0 = f'(0) = \xi(v) - F_{y^i}(y)v^i\xi(u) = \xi(v) - \mathbf{g}_y(y, v),$$

and the lemma is proved. □

Lemma 1.4 *Let (V, F) be a Minkowski space, then the dual norm F^* of F is a Minkowski norm on V^*, and for $\xi = l(y)$, we have*

$$g^{*ij}(\xi) := \frac{1}{2}\left[F^{*2}\right]_{\xi_i\xi_j}(\xi) = g^{ij}(y), \tag{1.14}$$

where $(g^{ij}(y)) = (g_{ij}(y))^{-1}$.

Proof It is obvious that F^* satisfies the conditions (1) and (2) in Definition 1.1. Notice that the matrix $(g^{ij}(y))$ is positive, in order to prove the lemma we need only to verify (1.14). Write

$$\xi = \xi_i \theta^i = l(y) = l(y^i \mathbf{b}_i),$$

then (1.10) yields $\xi_i = g_{ij} y^j$, and consequently,

$$\frac{\partial \xi_i}{\partial y^j}(y) = g_{ij}(y).$$

From (1.12) one gets $F^2(y) = F^{*2}(\xi)$, which together with above formula yield

$$\left[F^2\right]_{y^i}(y) = \left[F^{*2}\right]_{\xi_k}(\xi) g_{ki}(y). \tag{1.15}$$

We see from (1.15) and Euler's lemma that

$$g^{*kl}(\xi)\xi_l = \frac{1}{2}\left[F^{*2}\right]_{\xi_k}(\xi) = \frac{1}{2} g^{ik}(y)\left[F^2\right]_{y^i}(y) = y^k,$$

and thus

$$g^{*kl}(\xi)\xi_l \frac{\partial g_{ik}}{\partial y^j} = y^k \frac{\partial g_{ik}}{\partial y^j} = y^k \frac{\partial g_{ij}}{\partial y^k} = 0.$$

Taking partial derivative with respect to y^j for (1.15) we get

$$g_{ij}(y) = \frac{1}{2}\left[F^2\right]_{y^i y^j}(y) = \frac{1}{2}\left[F^{*2}\right]_{\xi_k \xi_l}(\xi) g_{ik}(y) g_{jl}(y) + \frac{1}{2}\left[F^{*2}\right]_{\xi_k}(\xi) \frac{\partial g_{ik}}{\partial y^j}(y)$$

$$= g^{*kl}(\xi) g_{ik}(y) g_{jl}(y) + g^{*kl}(\xi)\xi_l \frac{\partial g_{ik}}{\partial y^j}(y) = g^{*kl}(\xi) g_{ik}(y) g_{jl}(y),$$

thus (1.14) holds. □

By discussion above we know that the Legendre transformation $l : (V, F) \to (V^*, F^*)$ is a norm-preserving map. For $\xi \in V^* \backslash \{0\}$, denote by $\mathbf{g}^{*\xi}$ the fundamental form of V^* with respect to ξ, then

$$\mathbf{g}^{*\xi}(\zeta, \eta) = g^{*ij}(\xi)\zeta_i \eta_j, \quad \zeta = \zeta_i \theta^i, \quad \eta = \eta_i \theta^i.$$

It is clear from (1.10) and Lemma 1.4 that $y = l^{-1}(\xi)$ is determined by

$$\zeta(y) = \mathbf{g}^{*\xi}(\xi, \zeta), \quad \forall \zeta \in V^*, \tag{1.16}$$

namely,

$$y = l^{-1}(\xi) = y^k \mathbf{b}_k = g^{*ki}(\xi)\xi_i \mathbf{b}_k. \tag{1.17}$$

We can define the Legendre transformation $l^* : V^* \to V^{**}$ and the Minkowski norm F^{**} on V^{**} in a similar way. For any $y \in V$, let $y^{**}(\xi) = \xi(y), \forall \xi \in V^*$, then

$y^{**} \in V^{**}$. Identifying y and y^{**} in the canonical way, then $V = V^{**}, l^{-1} = l^*$, and $F = F^{**}$.

Example 1.4 Consider the Randers norm $F = \alpha + \beta$ on a vector space V, where α is an Euclidean norm, and β a linear functional on V with $\|\beta\|_\alpha = \sup_{\alpha(y)=1} \beta(y) < 1$. Let $\mathbf{b}_1, \cdots, \mathbf{b}_n$ be a basis of V with dual basis $\theta^1, \cdots, \theta^n$. Suppose that

$$\alpha(y) = \sqrt{a_{ij} y^i y^j}, \quad \beta(y) = b_i y^i, \quad y = y^i \mathbf{b}_i,$$

then $\|\beta\|_\alpha = \sqrt{a^{ij} b_i b_j}$, where $(a^{ij}) = (a_{ij})^{-1}$. It can be verified by the method of Lagrange multiplier that the dual Randers norm F^* on V^* is also a Randers norm, i.e., $F^* = \alpha^* + \beta^*$. Here the Euclidean norm α^* and the linear functional β^* on V^* can be expressed by

$$\alpha^*(\xi) = \sqrt{a^{*ij} \xi_i \xi_j}, \quad \beta^*(\xi) = b^{*i} \xi_i, \quad \xi = \xi_i \theta^i,$$

where

$$a^{*ij} = \frac{(1 - \|\beta\|_\alpha^2) a^{ij} + b^i b^j}{(1 - \|\beta\|_\alpha^2)^2},$$

$$b^{*i} = -\frac{b^i}{1 - \|\beta\|_\alpha^2},$$

and $b^i := a^{ij} b_j$. Let $(a_{*ij}) = (a^{*ij})^{-1}$, then

$$a_{*ij} = (1 - \|\beta\|_\alpha^2)\,(a_{ij} - b_i b_j).$$

Consequently, $\|\beta^*\|_{\alpha^*} := \sup_{\alpha^*(\xi)=1} \beta^*(\xi)$ satisfies

$$\begin{aligned} \|\beta^*\|_{\alpha^*}^2 &= a_{*ij} b^{*i} b^{*j} \\ &= \frac{1}{1 - \|\beta\|_\alpha^2}\,(a_{ij} - b_i b_j)\, b^i b^j = \|\beta\|^2, \end{aligned}$$

which shows that the lengthes of β and β^* with respect to α and α^* are equal. It is easy to know from (1.8) and (1.10) that the Legendre transformation $l : V \to V^*$ is determined by

$$\xi = l(y) = g_{ij}(y) y^j \theta^i = F(y) \left(\frac{a_{ij} y^j}{\alpha(y)} + b_i \right) \theta^i,$$

while the converse transformation is

$$y = l^{-1}(\xi) = g^{*kl}(\xi) \xi_l \mathbf{b}_k = F^*(\xi) \left(\frac{a^{*kl} \xi_l}{\alpha^*(\xi)} + b^{*k} \right) \mathbf{b}_k.$$