LECTURES ON ORDINARY DIFFERENTIAL EQUATIONS

Edited by
Robert McKelvey

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PREFACE

The lectures in this volume survey recent important tendencies in some of the active fields of research in ordinary differential equations. Taken together, they provide a comprehensive introduction to the subject, suitable for study in graduate seminars. They also provide a valuable resource for the active researchers in the field.

The paper of Lloyd Jackson surveys the subfunction method in two-point boundary value problems. Introduced originally by O. Perron and F. Riesz in studies of the Dirichlet problem, this method has emerged in the hands of Jackson, his students, and other workers as a comprehensive and systematic approach to a wide class of nonlinear, one-dimensional problems. In the lectures Jackson has reworked much of the subject, creating a unified and esthetically satisfying body of theory.

Both John Barrett and Douglas Willett deal with oscillation theory, though in quite different aspects. Willett systematizes and unifies a considerable body of recent results about the second order equation, often calling on ingenious technical devices to deal with highly complicated situations. Barrett's notes represent a pedigogical approach which he gradually refined in advanced seminars at the University of Utah and later at the University of Tennessee. By treating a series of prototype examples, Barrett gradually develops his reader's powers, finally approaching the difficulties inherent in the oscillation theory for equations of higher order than second.

It is the essence of singular perturbation theory that the perturbed boundary problem is a different kind of mathematical object from the unperturbed, a phenomenon associated with a change in order of the differential equation and an accompanying loss of boundary conditions. This intriguing aspect, combined with the pervasiveness in nature of asymptotically singular phenomena, has produced a voluminous literature, much of it having appeared since Wolfgang Wasow's well-known and now classical study, Robert O'Malley's article surveys the recent developments in this rapidly growing subject.

The articles in this volume are all related in one way or another to the symposium on ordinary differential equations held in Boulder in the summer of 1967. That symposium, the first of the Rocky Mountain Summer Seminars, was organized under the auspices of the Associated Rocky Mountain Universities, and financed by a grant from the Graduate Education Division of the National Science Foundation. The lectures of Barrett and Jackson were actually delivered at the symposium, in much the same form as here. Willett's survey represents an elaboration of an hour talk given at that time. O'Malley's notes, which had appeared earlier in mimeographed form, were the basis for one of several informal seminars held during the symposium.

It is especially fortunate that John Barrett retained the energy to develop through his symposium lectures and in these notes a definitive treatment of a subject which now bears his permanent mark. It was a privilege for me to have known this intelligent, considerate, and courageous man.

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Oscillation Theory of Ordinary Linear Differential Equations

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Preface

The purpose of this article [a compilation of lectures originally presented at the Associated Western Universities Differential Equations Symposium, Boulder, Colorado in the summer of 1967] is to give motivating examples and ideas which have influenced the author in his studies of oscillation properties of solutions of linear ordinary differential equations. This is a subject where the mathematical tools needed are relatively elementary but where it is easy to state an unsolved problem. For example, the familiar second-order equation $y'' + q(x)y = 0$ is still a valid subject for research, although it has a voluminous literature.

As far as oscillation theory is concerned, most texts in Differential Equations, both elementary and advanced, deal only with second-order equations. A few deal with self-adjoint fourth-order equations and, perhaps, those of arbitrary even order and systems of first or second-order

* We regret to report that Dr. Barrett died on January 21, 1969, and therefore that this paper is being published posthumously. We are grateful to Drs. John Bradley, William J. Coles, and John W. Heidel for reading the proofs.

equations. Any discussion of oscillatory properties of third-order equations or other nonself-adjoint equations is hard to find and that is the lowest order where truly nonself-adjoint equations occur.

In this article an attempt is made to give a self-contained inductive development from equations of one order to the next. Most of the discussion will deal with equations of second, third, and fourth orders, with linear systems of second-order equations and with generalizations of those results to equations of higher orders. No attempt is made to survey all of the oscillation theory of equations of orders higher than four. Considerable attention is devoted to equations of order three (Section II) and this is in line with the increased recent interest in these equations, as the Bibliography will show.

Instead of the usual format where proofs follow the respective theorems, motivating examples and developments of the ideas are given first with statements of theorems following as summaries of what has been established in the discussion.

I. Second-Order Equations

Much of the material in this introductory section is contained in introductory texts on Ordinary Differential Equations. Only those topics are included which are pertinent to the succeeding discussion of equations of higher order. For further oscillation theory of second-order equations see Chapters IV and XI of Hartman's recent advanced text (*47*).

1.1. *Basic Properties*

The real linear second-order equation

$$(1.1)\qquad \begin{aligned} &l_2[y] \equiv y'' + A(x)\,y' + B(x)\,y = 0;\\ &A \text{ and } B \in C(I), \qquad I = [a, b), \qquad a < b \leqslant \infty, \end{aligned}$$

is equivalent to a special case of the *canonical self-adjoint* form:

$$(E_2)\qquad L_2[y] = (ry')' + qy = 0; \qquad r > 0, \qquad r \,\&\, q \in C(I),$$

where $r = \exp(\int A)$ and $q = rB$. A function y is said to be *admissible for* the operator L_2 on an interval I provided y and $ry' \in C[I]$. Note that y'' need not exist when r is not differentiable. A *solution* of (E_2) is an admissible function for L_2 satisfying $L_2[y] = 0$ on I. Existence and uniqueness theorems for (E_2) may be obtained easily from the equivalent vector-matrix form

$$(V_2) \qquad \begin{pmatrix} y \\ D_1 y \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{r} \\ -q & 0 \end{pmatrix} \begin{pmatrix} y \\ D_1 y \end{pmatrix}; \qquad D_1 y = ry'.$$

Lemma 1.1. *For any pair of numbers* (c_0, c_1) *and each* $c \in I$ *there exists a unique solution of* (E_2) *satisfying*

$$y(c) = c_0, \qquad (ry')(c) = c_1.$$

There are several ways to transform (E_2) back into the form (1.1).

Lemma 1.2. (a) *If* $r \in C'(I)$ *then* (E_2) *can be put into the form* (1.1).
(b) *If* r *is not differentiable then the change of independent variable*:

$$(1.2) \qquad t = \int_a^x (1/r) \Rightarrow x = x(t), \qquad Y(t) = y[x(t)]$$

yields the form (1.1)—*with respect to* t—*without a middle term,*

$$\ddot{Y} + QY = 0, \qquad Q(t) = (qr)[x(t)].$$

Note that Lemma 1.2(b) provides a method for removing the middle term of (1.1) and differentiation of coefficients is not required, as it is in the standard variation of parameters substitution

$$y = uv, \qquad v' = (A/2)v.$$

Integration-by-parts of $zl_2[y]$ (or $zL_2[y]$), where z is an arbitrary function with the exhibited derivatives, yields useful *Lagrange Identities*.

Lemma 1.3. (a) *If* $y, z \in C''(I)$ *then*

$$zl_2[y] = \{zy' - yz' + Ayz\}' + yl_2^+[z],$$

where $l_2^+[z] = (z' - Az)' + Bz$, *an adjoint operator of* l_2.
(b) *If* z *and* y *are admissible functions for* L_2 *then*

$$zL_2[y] = [r(zy' - yz')]' + yL_2[z].$$

Note that $l_2^+ = l_2$ if $A \equiv 0$, and in this special case l_2 and (1.1) are *self-adjoint*. Since L_2 serves as its own adjoint operator, L_2 and the corresponding equation (E_2) are said to be *self-adjoint*.

1.2. *Factoring and Disconjugacy*

The Polya-Mammana (*73, 79*) factored form of L_2 is easy to establish and is actually a variation of standard Wronskian properties.

Theorem 1.1. *If* $L_2[v] = 0$ *and* $v \neq 0$ *on* $I' \subset I$ *then*

(a) $$vL_2[y] = [vD_1y - yD_1v]'$$

and

(1.4) $$L_2[y] = (1/v)[rv^2(y/v)']'$$

for each L_2*-admissible* y *on* I' *and*

(b) *no nontrivial solution of* (E_2) *has two zeros on* I'.

Definition 1.1. A *second-order* linear operator (e.g., L_2) and the corresponding homogeneous equation (e.g., E_2) are said to be *disconjugate* on an interval I provided that *no nontrivial solution* of the equation *has two zeros* on I.

Let $v_1 = v_1(x, a)$ be the solution of (E_2) defined by the initial conditions:

(1.3) $$y(a) = 0 \quad \text{and} \quad D_1y(a) = (ry')(a) = 1.$$

This solution is called the *principal* solution of (E_2) at $x = a$ and oscillation properties of (E_2) can be given in terms of $v_1(x, a)$.

The Polya–Mammana factored form (1.4) may be used to prove the following results which form a version of the Sturm comparison theorem.

Lemma 1.4. (a) *If* $a < b \leqslant \infty$ *then the equation* (E_2) *is disconjugate on* $I = [a, b)$ *if, and only if, the principal solution* $v_1(x, a) > 0$ *on* $I^0 = (a, b)$.

(b) *If* (E_2) *is disconjugate on an interval* I *then*

(E_2') $$(ry')' + q_1y = 0; \qquad q_1 \in C(I) \;\; \textit{and} \;\; q_1 \leqslant q \;\; \textit{on} \;\; I$$

is disconjugate on I.

(c) *If, in addition,* $r \leqslant r_1$ *and* $r_1 \in C(I)$ *then*

(E_2'') $$(r_1y')' + q_1y = 0$$

is disconjugate on I.

W. Leighton and Z. Nehari have isolated a crucial part of the standard proof of the Sturm Separation Theorem as a *Fundamental Lemma* and applied it to linear differential equations of orders greater than two (*70*).

Lemma 1.5. *If* $u(x)$ *and* $v(x)$ *are differentiable functions on* $[a, c]$, $a < c$, $u(a) = u(c) = 0$ *and* $v(x) \neq 0$ *on* $[a, c]$, *then*

(a) $vu' - uv' = v^2(u/v)'$ *and their Wronskian* $uv' - vu'$ *has a zero on* (a, c) *and*

(b) *there is a linear combination* $z = u - kv$ *which has a double zero on* (a, c) (*i.e., at* $x = \xi \in (a, c)$ *where* $z(\xi) = z'(\xi) = 0$).

Note that if u and v are also solutions of (E_2) and $a < c < b$ then they are linearly dependent, which contradicts the original assumptions of Lemma 1.5, thus yielding the Sturm separation theorem.

Lemma 1.4(a) provides a method for proving oscillation theorems, by use of nonlinear Riccati equations.

Lemma 1.6. *Let* $y(x)$ *be a solution of* (E_2) *on* $I = [a, \infty)$, *such that* $y(a) = 0$ *and* $y'(a) > 0$. *If* $y(x) > 0$ *on* (a, ∞) *then*

(a) $h = -ry'/y$ *satisfies a Riccati equation*

$$h' = q + (1/r)h^2 \quad on \quad (a, \infty). \tag{1.5}$$

(b) *If, in addition,* $y'(x) > 0$ *on* $[a, \infty)$ *then* $\int_a^x q$ *is bounded above on* $[a, \infty)$. *On the other hand, if* $\int_a^\infty q \equiv \infty$ *then* $y'(x)$ *has a zero on* (a, ∞) *and* $y(x)$ *is bounded on* $[a, \infty)$.

Once we have a zero of $y'(x)$ we proceed to force a subsequent zero of $y(x)$. Hille (*58*) seems to have been the first to note the following property, which was of considerable use to Nehari (*75*) and the author (*8*), for establishing necessary conditions for disconjugacy of (E_2).

Lemma 1.7. *Let* $y(x)$ *be a solution of* (E_2) *on* $I = [a, \infty)$ *such that* $y(a) > 0$, $y'(a) = 0$. *If* $y(x) > 0$ *and* $q(x) \geqslant 0$, *but* $\not\equiv 0$ *for large* x, *then* $D_1 y(x) = (r(x)\, y'(x)) \leqslant 0$ *on* (a, ∞) *and* $\int_a^\infty (1/r) < \infty$.

Theorem 1.2. *If* $\int_a^\infty (1/r) = \infty$, $q \geqslant 0$ *but* $\not\equiv 0$ *for large* x, (E_2) *is disconjugate on* $[a, \infty)$ *and* $y(x)$ *is any nontrivial solution of* (E_2) *with* $y(a) = 0$, *then* $y(x)\, y'(x) > 0$ *on* (a, ∞).

It is often useful to know that disconjugacy allows the construction of a nonzero solution on closed or open intervals (68). Recall that the converse (Lemma 1.4) is also true.

The case of the finite closed interval is easy to prove. On the other hand, if $I = [a, b)$, $a < b \leqslant \infty$, let $x_n \in (a, b)$ and $\{x_n\} \uparrow b$ and $y_n(x)$ be the unique solution of (E_2) such that

$$y_n(x_n) = 0, \qquad y_n(x) > 0 \quad \text{on} \quad [a, x_n) \quad \text{and} \quad (c_1{}^n)^2 + (c_2{}^n)^2 = 1,$$

where $y_n(x) = c_1{}^n u(x) + c_2{}^n v(x)$ and u, v is a given fundamental set of solutions of (E_2). There is a subsequence $\{n_j\}$ of $\{n\}$, such that $\{c_1^{n_j}\}$ and $\{c_2^{n_j}\}$ both converge, and these limits define a positive solution on $I^0 = (a, b)$, which may or may not be zero at $x = a$. Similarly, a positive solution may be found for open intervals I.

Theorem 1.3. *If (E_2) is disconjugate on an interval I then there exists a positive solution of (E_2) on I for* (a) $I = [a, b]$, $a < b < \infty$ *and* (b) $I = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$. (c) *If $I = [a, b)$ then there exists a positive solution on $I^0 = (a, b)$.*

1.3. *Oscillation*

By combining Lemma 1.6 with Theorem 1.2 we have:

Lemma 1.8. *If $\int_a^\infty 1/r = \infty$, $q \geqslant 0$ and $\int_a^\infty q = \infty$ then every solution of (E_2) has infinitely many zeros on $[a, b)$.*

This is a weak form of the Leighton–Wintner oscillation theorem. About 1949, both Leighton (*67*) and Wintner (*108*) eliminated the nonnegative condition, $q(x) \geqslant 0$. (See Theorem 1.2 below). However, the nonnegative coefficient case is of special interest and is more readily generalized to certain equations of higher order.

Definition 1.2. *A second-order operator L_2, or the corresponding equation (E_2), is said to be oscillatory {nonoscillatory} on an interval I provided that every solution of (E_2) has infinitely many {at most a finite number of} zeros on I.*

Although disconjugacy is an extreme case of nonoscillation they are essentially equivalent for second-order equations.

Lemma 1.9. *If L_2, or (E_2), is nonoscillatory on $[a, \infty)$ then there is a number $c \in (a, \infty)$ such that L_2, or (E_2), is disconjugate on $[c, \infty)$.*

Although it is well known that nonoscillation on $[a, \infty)$ implies disconjugacy for large x for second-order equations it was noted recently by Nehari (77) that the analogous statement for equations of orders greater than two is known to be true only for special cases. A useful *example* and comparison equation is *Euler's Equation*

$$x^2y'' + ky = 0, \qquad k = \text{constant}, \tag{1.6}$$

which

(i) has solutions of the form x^α where

$$\alpha \text{ is real if } \quad k \leqslant \tfrac{1}{4},$$
$$\alpha \text{ is complex if } \quad k > \tfrac{1}{4},$$

(ii) is disconjugate if $k \leqslant \frac{1}{4}$ and oscillatory if $k > \frac{1}{4}$ on $[1, \infty)$.

However, when (1.6) is put into the form (E_2), with $r = 1$ and $q = k/x^2$, it *does not satisfy the hypothesis of Lemma 1.8* on $[1, \infty)$ since $\int_1^\infty q < \infty$ for all values of k, although we have oscillation for $k > \frac{1}{4}$. The following is well-known.

Lemma 1.10. *If $\int_a^\infty (1/r) < \infty$ and $\int_a^\infty |q| < \infty$ then equation (E_2) is nonoscillatory on $[a, \infty)$.*

Suppose that (E_2) is nonoscillatory, i.e., there exists a solution $u(x)$ and a number $b \in (a, \infty)$ such that $u(x) > 0$ on $[b, \infty)$. Therefore, $h = -ru'/u$ satisfies the *Riccati Equation*

$$h' = q + \frac{1}{r} h^2 \quad \text{on} \quad [b, \infty). \tag{1.5}$$

Let $\int_a^\infty q = \infty$ and $a < b < \infty$, then there is a number $c \in (b, \infty)$ such that $h(b) + \int_b^x q > 0$ on $[c, \infty)$ and, hence,

$$h(x) > g(x) = \int_b^x (1/r)\, h^2 > 0 \quad \text{on} \quad [c, \infty).$$

Consequently,

$$g' > (1/r)\, g^2 \qquad \text{and} \qquad \int_c^x (1/r) < \frac{1}{g(c)} < \infty.$$

This proof for the following Leighton–Wintner Oscillation Theorem was suggested by W. J. Coles.

Theorem 1.4. *If* $\int_a^\infty 1/r = \infty$ *and* $\int_a^\infty q = \infty$ *then the equation* (E_2) *is oscillatory on* $[a, \infty)$.

This theorem is the *principal motivation* for the discussion in the subsequent chapters where higher-order analogs are established. Willet (*107*) has recently pointed out that if $\int_a^\infty (1/r) < \infty$ then the transformation

$$t = \left[\int_x^\infty (1/r)\right]^{-1} \tag{1.6}$$

yields the following corollary of Theorem 1.4.

Corollary 1.4.1. *If* $\int_a^\infty (1/r) < \infty$ *but*

$$\int_a^\infty q(s)\left[\int_s^\infty (1/r)\right]^2 ds = \infty \tag{1.7}$$

then (E_2) *is oscillatory on* $[a, \infty)$.

In the case of (E_2), with $r = 1$ and $q \geqslant 0$, Hille (*58*) achieved better results than Lemma 1.8. Let $\int_a^\infty (1/r) = \infty$, $q \geqslant 0$ but $\neq 0$ for large x and (E_2) be disconjugate on $[a, \infty)$ and let $y(x)$ be a *positive solution* (E_2) on (a, ∞). Then, by Theorem 1.2, $h = -ry'/y < 0$ on (a, ∞). Since h satisfies the Riccati equation (1.5), it follows that

$$|h(x)| = -h(x) \geqslant \int_x^\infty q \qquad \text{and} \qquad \int_a^x (1/r) \leqslant -1/h(x).$$

Theorem 1.5. *If* $\int_a^\infty (1/r) = \infty$, $q(x) \geqslant 0$ *but* $\neq 0$ *for large* x *on* $[a, \infty)$ *then a necessary condition for disconjugacy of* (E_2) *on* $[a, \infty)$ *is*

$$\int_a^x (1/r) \int_x^\infty q \leqslant 1. \tag{1.8}$$

Hence, $\limsup_{x\to\infty} \int_a^x (1/r) \int_x^\infty q > 1$ *is sufficient for oscillation of* (E_2).

1.4. *The Prüfer Transformation*

The change of variables to polar coordinates of a nontrivial solution $y(x)$ of Eq. (E_2) in its phase plane

$$(1.9) \qquad \begin{cases} y(x) = \rho(x) \sin \theta(x), \\ D_1 y(x) = \rho(x) \cos \theta(x); \end{cases} \qquad D_1 y = r y',$$

is called a *Prüfer Transformation*, after H. Prüfer who introduced the idea in 1926 (*80*). The approach given here is due to W. T. Reid (*87*) and it is convenient for generalization to higher order self-adjoint systems.

Suppose that $y(x)$ is a nontrivial solution of (E_2) and let

$$(1.10) \qquad \rho(x) = \sqrt{y^2(x) + (D_1 y(x))^2} > 0.$$

Next normalize y and $D_1 y$ by letting

$$(1.11) \qquad s(x) = y(x)/\rho(x) \quad \text{and} \quad s_1(x) = D_1 y(x)/\rho(x).$$

By differentiating (1.10) and (1.11) we have

$$(1.12) \qquad \rho'/\rho = (1/r - q)\, s s_1$$

and

$$(1.13) \qquad \begin{pmatrix} s \\ s_1 \end{pmatrix}' = \begin{pmatrix} 0 & b(x) \\ -b(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ s_1 \end{pmatrix}; \qquad b(x) = s_1{}^2(x)/r(x) + q(x)\, s^2(x).$$

Therefore, if $\theta'(x) = b(x)$ then (1.9) is fulfilled and the Prüfer differential equations for (E_2) are

$$(1.14) \qquad \begin{cases} \text{(a)} \quad \rho' = (1/2)[(1/r - q) \sin 2\theta]\rho, \\ \text{(b)} \quad \theta' = (1/r) \cos^2 \theta + q \sin^2 \theta \\ \qquad\quad = (1/2)(1/r + q) + (1/2)(1/r - q) \cos 2\theta. \end{cases}$$

An alternate approach is to show that the nonlinear θ-equation of (1.14) has a unique solution for each given initial value of θ at some $a \in I$ and then to solve the other (linear) equation for ρ. Also, (1.14) may be derived by differentiating (1.9).

Theorem 1.6. *Each solution of equation* (E_2) *may be expressed by* (1.9) *whose polar components satisfy* (1.14).

Observe that if $\theta(x)$ is a solution of (1.14b) and $\theta(b) = k\pi$ then

$$\theta'(b) = 1/r(b) > 0.$$

Therefore, even though $\theta(x)$ may not be monotone it is always increasing at multiples of π, which has an important bearing on oscillation properties.