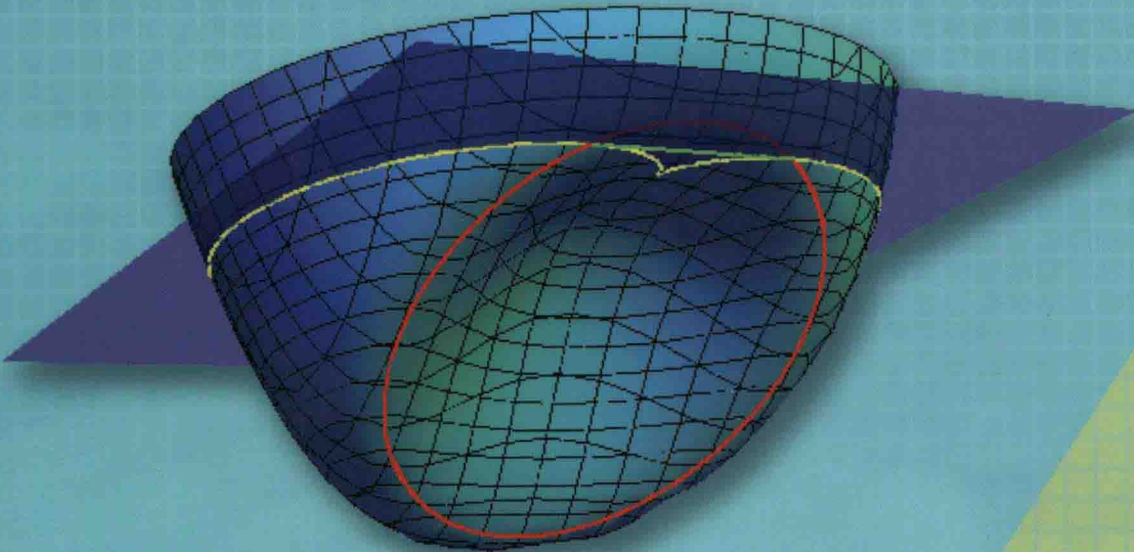
SEMIDEFINITE OPTIMIZATION *and* CONVEX ALGEBRAIC GEOMETRY



Edited by
Grigoriy Blekherman
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MOS-SIAM Series on Optimization

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MOS-SIAM Series on Optimization

This series is published jointly by the Mathematical Optimization Society and the Society for Industrial and Applied Mathematics. It includes research monographs, books on applications, textbooks at all levels, and tutorials. Besides being of high scientific quality, books in the series must advance the understanding and practice of optimization. They must also be written clearly and at an appropriate level for the intended audience.

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Preface

In the past decade there has been a surge of interest in algebraic approaches to optimization problems defined in terms of multivariate polynomials. Fundamental mathematical challenges that arise in this program include understanding the structure of nonnegative polynomials, the interplay between efficiency and complexity of different representations of algebraic sets, and the development of effective algorithms. Remarkably, and perhaps unexpectedly, convexity provides a new viewpoint and a powerful framework for addressing these questions. This naturally brings us to the intersection of *algebraic geometry*, *optimization*, and *convex geometry*, with an emphasis on algorithms and computation. This emerging area has become known as *convex algebraic geometry*.

Our aim is to provide an accessible and unifying introduction to the many facets of this fast-growing interdisciplinary area. Each chapter addresses a fundamental aspect of convex algebraic geometry, ranging from the well-established core mathematical theory to the forefront of current research and open questions. Throughout we showcase the rich interactions between theory and applications.

This book is suitable as a textbook in a graduate course in mathematics and engineering. The chapters make connections to several areas of pure and applied mathematics and contain exercises at many levels, providing multiple entry points for readers with varied backgrounds.

We thank the National Science Foundation for funding a Focused Research Group grant (2008–2011) awarded to Bill Helton, Jiawang Nie, Pablo A. Parrilo, Bernd Sturmfels, and Rekha R. Thomas. This award enabled a flurry of research activity in semidefinite optimization and convex algebraic geometry. Several workshops and conferences were organized under this grant's support. In particular this book was inspired by the lectures at the workshop LMIPO organized by Bill Helton and Jiawang Nie at the University of California, San Diego in March 2010.

We thank all our contributors for their hard work and perseverance through multiple rounds of edits. We also thank Tom Liebling, Sara Murphy, and Ann Manning Allen at SIAM for their support and patience with the production of this book. Special thanks to our students and colleagues who read versions of this book and sent us comments, in particular Chris Aholt, Hamza Fawzi, Fabiana Ferracina, Alexander Fuchs, Chris Jordan-Squire, Frank Permenter, James Pfeiffer, Stefan

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List of Notation

Basics:	
fields, rings	$\mathbb{R}, \mathbb{C}, \mathbb{P}, \mathbb{Q}, \mathbb{Z}$
nonnegative integers	$\mathbb{N}$
nonnegative orthant	$\mathbb{R}^n_+$
positive orthant	$\mathbb{R}^n_{++}$
standard simplex in $\mathbb{R}^n_+$	$\Delta_n := \{x \in \mathbb{R}^n_+ : \sum x_i = 1\}$
standard basis vectors	e_i
Matrices:	
$m \times n$ matrices	$\mathbb{R}^{m \times n}$
matrix brackets	[ ]
$n \times n$ symmetric matrices	$\mathcal{S}^n$
$n \times n$ positive semidefinite definite matrices	$\mathcal{S}^n_+$
$n \times n$ positive definite matrices	$\mathcal{S}^n_{++}$
inner product in $\mathcal{S}^n$	$\langle A, B \rangle$
matrix multiplication	$A \cdot B$
trace	Tr
matrix transpose	A^T
determinant	$\det M$
rank	$\operatorname{rank} M$
diagonal of a matrix M as a vector	$\operatorname{diag}(M)$
diagonal matrix obtained from a matrix M	$\operatorname{Diag}(M)$
lower triangular matrix from matrix M	$\operatorname{Tril}(M)$
turning a vector v into a diagonal matrix	$\operatorname{Diag}(v)$
block diagonal matrix with blocks A, B etc	$\operatorname{BlockDiag}(A, B, ...)$
positive semidefinite	$\succeq 0$
positive definite	$\succ 0$
max/min eigen/singular value	$\lambda_{\max}, \sigma_{\min}$
Geometry:	
p-norm	$\|u\|_p$
ball with center u, radius r	$B(u, r)$
vector space dual	V^*
orthogonal complement of vector space	$V^\perp$
dimension	$\dim V$

codimension	$\operatorname{codim} V$
cone dual	C^*
polar dual of convex body	P°
dual face to an exposed face	$F^\diamond$
dual variety	X^*
interior of a set	$\operatorname{int}(C)$
boundary of set	∂C
algebraic boundary	$\partial_a C$
closure of set	$\operatorname{cl}(C)$ or $\overline{C}$
convex hull of set C	$\operatorname{conv}(C)$
conical hull of set C	$\operatorname{cone}(C)$
gauge function of a convex body K	$G_K(x)$

Optimization:

optimal solution	$u^\star$
semidefinite program	SDP
kth theta body of ideal I	$\mathrm{TH}_k(I)$
characteristic vector of a set S	χ^S

Algebra:

ideal generated by	$\langle f_1, \ldots, f_m \rangle$
variety of ideal	$V_{\mathbb{R}}(I), V_{\mathbb{C}}(I)$
vanishing ideal of a set	$I(S)$
Jacobian	$\mathrm{Jac}(\ )$
gradient	∇
Hessian	∇^2
singular locus	$\mathrm{Sing}(\ )$
smooth points in a variety	X_{reg}
polynomial ring in n variables	$\mathbb{R}[x], \mathbb{C}[x]$
polynomials in n variables, degree at most d	$\mathbb{R}[x]_{n,d}$
if n clear	$\mathbb{R}[x]_d$
monomials of degree at most d	$[x]_d$
$\alpha \in \mathbb{N}^n$ (for exponents of monomials)	$\lvert\alpha\rvert = \sum \alpha_i$
nonnegative polynomials in n variables, degree at most $2d$	$P_{n,2d}$
if n is clear	P_{2d}
sum of squares in n variables of degree at most $2d$	$\Sigma_{n,2d}$
if n is clear	Σ_{2d}
forms in n variables, degree equal to d	$\mathbb{R}[x]_{\mathbf{n,d}}$
if n clear	$\mathbb{R}[x]_{\mathbf{d}}$
monomials of degree d	$[x]_{\mathbf{d}}$
nonnegative forms in n variables, degree $2d$	$P_{\mathbf{n,2d}}$
if n is clear	$P_{\mathbf{2d}}$
sos forms in n variables of degree $2d$	$\Sigma_{\mathbf{n,2d}}$
if n is clear	$\Sigma_{\mathbf{2d}}$

sos polynomials mod an ideal I	$\Sigma(I)$
polynomials in $\mathbb{R}[x]_{n,d}$ that are k-sos mod I	$\Sigma^k_{n,d}(I)$
if n is clear	$\Sigma^k_d(I)$
affine linear polynomials in above	$\Sigma^k_1(I)$
Newton polytope of f	$\mathcal{N}(f)$
linear functionals on $\mathbb{R}[x]$	ℓ
linear functionals that are evaluations at v	ℓ_v
quadratic forms on $\mathbb{R}[x]_{\mathbf{n},\mathbf{d}}$	$S^{n,d}$
nonnegative quadratic forms in $S^{n,d}$	$S^{n,d}_+$
preorder of $g_1, \ldots, g_m$/truncated	$\mathbf{preorder}(g_1, \ldots, g_m)$, $\mathbf{preorder}_k(g_1, \ldots, g_m)$
quadratic module of $g_1, \ldots, g_m$/truncated	$\mathbf{qmodule}(g_1, \ldots, g_m)$, $\mathbf{qmodule}_k(g_1, \ldots, g_m)$

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