Operations Research Methods

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||*Mangalacarnaṁ*||

kiỳm karma kimakarmeti kavayo'pyatra mohitāḥ |
tatte karma pravakṣyāmi yajjñātvā mokṣyase'śubhāt ||

karmaṇo hyapi bodhavyaṁ bodhavyaṁ ca vikarmaṇaḥ |
akarmaṇaśca bodhavyaṁ gahanā karmaṇo gataḥ ||

gatasaṅgasya muktasya jñānāvasthitacetasaḥ |
yajñāyācartaḥ karma samagraṁ pravilīyate ||

udārāḥ sarva evaite jñānī tvātmaiva me matam |
āsthitaḥ sa hi yuktātmā māmevānuttamām gatim ||

—- **From Śrīmad Bhagvadgītā**

PREFACE

In the spirit of a mathematical science, Operations Research is spread over many applied disciplines such as, Engineering, Management, Economics, etc. Topics from the subject in varying details, form part of courses in these disciplines at the graduate and sometimes at postgraduate levels. The purpose is to impart knowledge of quantitative decision-making techniques which crop up in an organisational set-up. The methods bewilderingly vary, though there is uniform desire for achieving economy and efficiency. As a result books on the subject tend to be bulky. For this book, I have set forth the objective of treating most of the important methods in a reasonable volume which may meet preliminary requirement of students belonging to various disciplines.

The disciplines impart special world view and the students may prefer to learn the subject from their viewpoint. The core of the methods of operations research is, however, mathematical in nature no matter what the setting is, and this point of view is constantly kept in mind while developing the array of methods. This is achieved after completely proving the component parts. This enables enunciation of the methods as algorithms that may, if required, be converted into computer codes for general–purpose implementation. I have not ventured into this aspect, regarding it as beyond scope of the book. As for application of each method, there exist beautiful bouquets from various disciplines. These have been incorporated in the illustrative examples and many more in the adjoining exercises. I hope the students would enjoy working them out and gain deeper insight of each method. The complete mathematical proofs, I hope will impart full understanding and satisfaction. At the same time, the mathematical aspects have been kept to the extent necessary, saving boredom to those not mathematically inclined. Elementary knowledge of matrix algebra including partition of matrices and probability theory is presumed.

The book consists of eight chapters as detailed in the *Contents*. The short opening chapter describes the nature of operations research problems in general terms. The second chapter on Linear Programming (there is increasing evidence in mathematical circles to prefer the term “optimisation” instead of the historic “programming”) is lengthy in view of the issues involved. The remaining five chapters on Transportation and Assignment, Networks, Integer and Dynamic Programming (!), Decision Analysis and Queues are of reasonable length. The last chapter introduces simulation approach to the subject.

I hope that the students would find coverage of material (*models* in technical terminology) adequate for their purpose. Many may sorely miss Karmarkar's method in Linear Programming and some other topics, but the length devoted to linear programming prevented inclusion. In other cases, I have felt that students can consider such topics as material for further studies. A large body of case studies and material of mathematical nature exists in literature which an aspiring student may explore.

In writing this book, my thanks go to Professor P.K. Das and the succeeding Heads of the Department of Computer Science, Jadavpur University, Kolkata for encouraging me to test the deep ethereal waters of operations research. Though impossible, I felt like having it *fathomed* when M/s Narosa Publishing House agreed to publish the book. My sincere thanks goes to them.

Sujit K Bose

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CHAPTER 1

GENERAL FEATURES OF OPERATIONS RESEARCH

The origins of the subject of **Operations Research** often abbreviated as **O.R.**, date back to the second world war (1939 – 1945) of the last century. During the war, the allies U.K. and U.S.A. engaged groups of scientists belonging to different disciplines, in order to urgently evolve scientific approach to methodologies for allocation of scarce resources required for different activities in drawing up effective tactical and strategic operational plans. It is claimed that the recommendations of these groups led to victories in several theatres of the war.

With the end of the war in the year 1945, these countries witnessed industrial boom with the aid of newly invented technologies. The growing industries facing increasing complexities of their activities had to seek consultancy from such scientists who had worked in O.R. groups during the war. This was due to gradual awareness that industrial and business activities, in essence, were of similar nature as were encountered in the military operations. Actually during this period the subject of Operations Research acquired definitive shape. Beginning with the pioneering work of the mathematician G.B.Dantzig during 1947 – 1948, most of the methods of O.R. were in place by the end of the next decade. In succeeding decades intensive development of the subject has taken place with the discovery of newer more efficient methods, treating large size activities and development of computer codes. Applications in new areas of agriculture, hospitals, communication networks and government have also been found.

The interdependence of economic activities, probably was first stressed long ago in 1758 by François Quenay in a memoir entitled *Tableau économique.* In 1955, A. Phillips showed that the Tableau économique could be converted in to what is known as the Leontief's input – output model in economic theory [see chapter 2, section 1.1(v)]. It is likely that following this French title, the French term *Tableau* occurs commonly in O.R. rather than *Table* in English.

Operations Research is thus applicable to a variety of **organisations.** An organisation often has to deal with choice among alternative **activities** arising from conflict of interest among its own components as well as those arising from other organisations with which it has dealings.

In some cases the choice has to be made in the backdrop of uncertainties. Thus the subject of O.R. is really "research" on operations or **activities** for most effectively running the organisation. Due to diverse nature of organisations it is evident that unlike pure sciences where answers can be deduced from a few fundamental laws, the answers in O.R. are obtained in a variety of ways. Nevertheless, with the development of the subject, some standard types have emerged and the answers in the first instance may well be found in these types. The scope of modification and improvement is of course always there. When uncertainties are supposed absent the standard types are called **deterministic**, while in the contrary case they are called **probabilistic**. The standard types have the common running theme of obtaining most effective (often called **optimal**) solution for the activities of the organisation.

In order to critically understand the activities of an organisation disregarding those in the fringe (in other words performing **Systems Analysis** of the organisation) and come up with O.R. answers, a **team** of individuals having specialised skills in different disciplines is required. Typically an O.R. team may contain those who are highly trained in O.R. methods, mathematics, probability and statistical theory, applied statistics, economics, business administration, computing, production, construction, process and communication engineering, agriculture, hospital management and physical sciences. Thus **teamwork** is a necessary ingredient of operations research. The role of the team is **advisory** to the management of the organisation and must come up with their specific recommendations, including alternatives, but decisions are taken only by the management. It is clear that some members of the team must be in good communication with the management before beginning the exercise and at the end present their recommendations.

When an organisation is very large, an O.R. team is usually engaged to find solution for critically important segments of the organisation, which otherwise is not apparent. From the recommended solutions, including alternatives, the management after taking into consideration the remaining segments of the whole organisation decides on the course of the activities, keeping in view the overall objective of stable profit, increased market share, product diversification, stable prices, improved worker morale, company prestige, social responsibilities while maintaining family control.

Because of its very nature, the management communicates to the O.R. team in a vague, imprecise language about the segment requiring O.R. solution. From the input, the team performs **Systems Analysis**, this means that it has to identify the important **activities** involved, together with the associated **alternative decisions** which could be made and the **appropriate objectives** to be attained. For quantitative answers it is apparent that **mathematical modeling** of the system is required in which the essence of the above mentioned ingredients are taken into account. This procedure is true of most of pure sciences as well, but the difference is that the O.R. models are far more inexact because of the inherent complexities of the system, losing minute details during abstract modeling.

Suppose that the identified **decision alternatives** are n in number say $x_1, x_2, ..., x_n$ (often the mathematical terminology **decision variables** is used for these alternatives). These are often quantitative in nature and subject to **constraints** which can be formulated as mathematical inequalities. The **objectives** obviously depend on $x_1, x_2, ..., x_n$ and are, therefore, mathematical functions of these variables. For O.R. solution the values of $x_1, x_2, ..., x_n$ are required to be determined for which the **objective functions** are optimal, that is to say, maximum or minimum. Thus a model may look like

Optimise: Objective Functions of $x_1, x_2, ..., x_n$
s.t.: Constraints involving $x_1, x_2, ..., x_n$

(s.t. standing for "subject to"). In this book, we shall restrict to models in which there is only one objective function.

The solution methods are generally called **programming** (not to be confused with *computer programming*) by which actually **planning of activities** is implied. A well known class of models are the **Linear Programming Models** in which the objective function and the constraints are all linear functions of $x_1, x_2, ..., x_n$. In the following chapters, we shall come across several models belonging to this class. When either the objective function or any of the constraints is a nonlinear function of $x_1, x_2, ..., x_n$, the model is called **Nonlinear Programming Model**. In this book we shall occasionally come across such models.

The models alluded above generally contain several parameters associated with the decision alternatives. These are supposed to be estimated from the given system. If taken as **constants**, the models are **deterministic**. The effect of slight alterations in them due to some uncertainty is an important question and is called **Sensitivity Analysis**. An O.R. team is required to perform this analysis too and incorporate appropriate recommendations in their report. In some other models of a different kind, the decision alternatives depend on random parameters with some probabilities (which is not unnatural because of complexity of systems) and so the models become **probabilistic**. In such cases the treatment is based on the probability theory in which **expectation** or **mean value** plays a central role. A direct and more important class of probabilistic model is that of systems operating over a length of time, receiving and despatching items at random. Historically, the mathematical theory of such problems was developed by noted mathematical probabilists dating back to the beginning of the last century. Enriched by application models the theories were later on embodied in O.R.. In this connection it will not be out of place to mention that the literature on the mathematics of the variety of problems constituting the deterministic and probababilistic models is extremely rich.

Over the years several models have been standardised, some of which form the basis of this book. But the application of a particular model to an organisational setup requires additional task of **model validation**, it being presumed that gross errors in model building and trivial errors of computation are not present. A systematic approach when applicable, is the **retrospective test**. In this test, past data for the model parameters are used and the solution compared with past performance. If the solution is indicative of a degree of agreement with the experience of possible good performance in the past, then the model can be applied in the current situation as well, provided the system behaviour has not radically changed. In some cases past data may not be available. In such cases *status quo* in the level of activities can be maintained for some time, to generate data and the performance compared with the solution. In case of major disagreement, the scope of improvement in the model structure opens and the O.R. team should be in a position to address such eventuality.

Over a length of time, due to complexity of real world, system parameters may undergo change and these require constant monitoring. The solution computer programs or special software packages, based on them must have flexibility to make changes in these parameters so that repeated application is possible.

Implementation of the solution of a validated model is the final task of an O.R. team. In liaison with the administrator and his/her personnel, the team is required to document **Operating Instructions** in readily understandable form, the solution or solutions under change of parameters, to initiate the desired course of activities.

CHAPTER 2

LINEAR PROGRAMMING

1. Linear Program and the Product Mix Model

Many Operations Rsearch mathematical models have the characteristic feature that the Objective Function is some **linear** function of a certain number of n decision alternatives $x_1, x_2, ..., x_n$ which are subject to a certain number of **linear** constraints. Alternative resources to $x_1, x_2, .., x_n$ at different levels satisfying the constraints results in different values of the objective function and the problem is to determine the levels for which the objective function becomes optimal (maximal or minimal).

Historically, the problem in its general form was conceived by the mathematician G.B. Dantzig and his associates in 1947, while working for United States Department of Air Force. Beginning from the same year the economist T.C. Koopmans discovered that the formulation provided an excellent framework for the analysis of classical economic theories. The term **Linear Programming** was coined by Koopmans and Dantzig in the summer of 1948 during the course of their respective work. Dantzig's group working on project SCOOP (Scientific Computation of Optimum Programs) developed the theory and a range of applications. In 1949 Dantzig gave the **Simplex Method** of solution of the problem and put in place most of the theory. The development of the subject in the middle of twentieth century is considered a landmark in the history of mathematics. The theory is however predated by some related work in 1939. L.V. Kantorovich considered a restricted class of problem occurring in the modeling of **production planning** and proposed a rudimentary method of solution. His work however remained neglected in the Soviet Union and outside, but was much later awarded the Nobel Prize in Economics with Koopmans in 1975. In the same year (1939), W. Karush investigated some aspects of optimality of a function of n variables subject to inequality constraints. These discoveries have proved useful in linear programming theory. In subsequent years substantial developments have taken place; notably by the discoveries of N. Karmarkar (1984). Adopting a radically different approach now called *Interior Point Method*, substantial reduction in computing time has been achieved for large dimensional problems.

In building a **linear programming** (or briefly **LP**) **model** for a system, **two** crucial assumptions must be valid. The *first* is that the activity level x_j $(j = 1, 2, ..., n)$ has a **proportional** contribution to the **objective function** as well as in a **constraint**. If $c_j x_j$ and $a_{ij} x_j$ are the respective contributions in the objective and the i^{th} constraint, the **per unit** contributions c_j and a_{ij} are assumed constants for the system and must be known from analysis of the actual system. The proportionality assumption precludes economies of return to scale or **discounts**. The *second* assumption is that of **additivity** of the contributions. The system must be such that the objective function and each of the constraint can be obtained from the contributions by *simple addition*. In this way, the objective function and each of the constraints become **linear functions** of $x_1, x_2, ..., x_n$.

As an example, consider the **Product Mix model**, applicable for **production planning**. Suppose that a production unit (or **firm** in the terminology of economics) produces n number of products utilising m number of **resources** (of its own or of others) having *maximum available* **capacities** $b_1, b_2, ..., b_m$. Let a_{ij} be the amount consumed from the i^{th} source for the production of a *unit* of the j^{th} product which ensures **return** (or profit) c_j. These data are supposed known from the cost analysis for the production unit and can be arranged as a table in the following manner:

Resource	Product				Available Capacity
	1	2	$\cdots$	n	
1	a_{11}	a_{12}	$\cdots$	a_{1n}	b_1
2	a_{21}	a_{22}	$\cdots$	a_{2n}	b_2
.	.	.	$\cdots$	.	.
.	.	.	$\cdots$	.	.
m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	b_m
Unit Return	c_1	c_2	$\cdots$	c_n	

Given the data it is required to find the level of production of the n products so that the total return z is maximum. If we suppose that the level of production is x_j for the j^{th} product $(j = 1, 2, ..., n)$, then the mathematical problem of determining these unknowns becomes the **Linear Program** (also written again in short as **LP**):

$$
\begin{aligned}
\text{Maximise:} \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t:} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
& x_1, x_2, ..., x_n \ge 0.
\end{aligned}
$$

If the system allows $x_1, x_2, ..., x_n$ to be *divisible* numbers then their values could be integer, fractional and even theoretically any real number. This is obviously the case for example, a paints manufacturer producing paints of different colours and shades. There are however several instances, like manufacture of television sets, music systems etc. of different models in an electronic manufacturing company, where it is required that $x_1, x_2, ..., x_n$ must be integers, then this constraint also enters the list of constraints written above. The problem then becomes an Integer LP problem and is studied separately in chapter 5. In this chapter we shall treat the former case.

In general, instead of manufacture of certain products one can have certain other **activities** of an organisation that can be modeled on the linearity assumption. Evidently the mathematical problem remains the same.

Remark 1. In many cases the available capacities are not simple constants but depend on *set–up* and *take–down* time with associated costs. In such cases the above model can not be used.

In the next subsection we consider several other examples of LP modeling.

1.1. Additional LP Models

(i) Diet Model. This model is one of the earliest, developed by G.J. Stigler in 1945. An animal farm (say a dairy farm) requires n different animal feeds (straw, green grass, gram etc.) available at prices $c_1, c_2, ..., c_n$ *per unit* of weight (kg). The minimum daily requirement by weight of essential m nutrients (carbohydrate, protein, vitamins etc.) are suppose $b_1, b_2, ..., b_m$, for best yield. Nutrition analysis shows that the j^{th} feed *per unit* by weight contains a_{ij} amount of the i^{th} nutrient. It is required to find, amounts of feed $x_1, x_2, ..., x_n$ required at minimum cost z consistent with the nutrition levels. The problem in compact form is thus:

$$\text{Minimise: } z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.: } \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad x_j \geq 0$$

Remark 2. In O.R. whenever it is a question of **cost** in the objective function, it is **minimised**. On the other hand when it is a question of **return** (profit), it is **maximised**.

(ii) Land Use Model. A farmer has h hectares of land in which he wants to sow n crops (say, wheat, sugar cane, mustard, vegetables etc.) from which he expects returns of $c_1, c_2, ..., c_n$ per hectare. The water requirement per hectare are respectively $w_1, w_2, ..., w_n$ units of which the maximum availability is w units. Similarly the fertilser and pesticide requirements per hectare are $f_1, f_2, ..., f_n$ and $p_1, p_2, ..., p_n$ with maximum availability of f and p units respectively. The farmer wants to allocate more acreage to the first crop than the rest combined together. It is required to find $x_1, x_2, ..., x_n$ hectares which he should sow for maximum return z. The problem is thus:

$$\text{Maximise}: \quad z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.}: \quad \sum_{j=1}^{n} x_j = h, \quad \sum_{j=1}^{n} w_j x_j \le w, \quad \sum_{j=1}^{n} f_j x_j \le f, \quad \sum_{j=1}^{n} p_j x_j \le p,$$

$$x_1 - (x_2 + x_3 + \cdots + x_n) > 0,$$

$$x_1, x_2, ..., x_n \ge 0.$$

It is evident that there can be variants in the constraint for acreage allocated to different crops on the choice of the farmer.

(iii) Bank Loan Model. A bank is in the process of devising a new policy of giving loans to four sectors: Commercial, Personal, Home and Car to an individual. The total capital is l (in Rupees or some other currency) and the bank estimates that it can charge interest at rates r_1, r_2, r_3, r_4 respectively in the four sectors. From previous experience the bank knows that the probability of recoverable bad debts in the four sectors will be p_1, p_2, p_3, p_4 (which are usually small fractions). For viability of the policy the bank sets at least 40% of the total capital for commercial loans and to encourage housing sector it sets that the home loans must be at least 50% of the personal and car loans. On bad debts the bank's policy is that the total must not exceed 5% of the total loans given. It is required to find the optimal investments x_1, x_2, x_3, x_4 in the four sectors.

The objective function z is given by

$$z = \sum_{j=1}^{4} r_j(1 - p_j)x_j - \sum_{j=1}^{4} p_j x_j.$$

Writing the constraints successively, the problem becomes

$$\text{Maximise}: \quad z = \sum_{j=1}^{4} [r_j - p_j(1 + r_j)]x_j$$

$$\text{s.t.}: \quad \sum_{j=1}^{4} x_j \le l, \quad x_1 \ge 0.4l, \quad x_3 \ge 0.5(x_2 + x_4), \quad \sum_{j=1}^{4} p_j x_j \le 0.05 \sum_{j=1}^{4} x_j,$$

$$x_1, x_2, x_3, x_4 \ge 0.$$