EXISTENCE AND REGULARITY OF MINIMAL SURFACES ON RIEMANNIAN MANIFOLDS

BY

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For my parents, Bishop and Mabel Pitts,
and my wife, Karen, whose love and
patience have been my inspiration

CONTENTS

INTRODUCTION

In this monograph, we develop a comprehensive variational calculus with which we explore the existence and regularity of minimal surfaces on riemannian manifolds. Our principal conclusion is the following theorem.

> THEOREM A. EXISTENCE THEOREM FOR REGULAR MINIMAL HYPERSURFACES ON RIEMANNIAN MANIFOLDS (7.13).
> *If* $2 \le k \le 5$, $\max\{k,4\} \le \nu \le \infty$, *and* M *is a* (k+1)-*dimensional compact riemannian manifold of class* $(\nu+1)$, *then* M *supports a nonempty, compact,* k *dimensional, imbedded, minimal submanifold (without boundary) of class* ν.

In these dimensions this theorem answers completely a more general question; namely, for what positive integers k and n does a smooth, compact, n dimensional, riemannian manifold support a regular closed minimal submanifold of dimension k ? Classically the only case in which there were satisfactory answers of great generality was when $k = 1$ and n was arbitrary (existence of closed geodesics). The first breakthrough to higher dimensions without severe restrictions on the ambient manifold came in 1974 when we established a precursor of theorem A, valid when $k = 2$ and $n = 3$. (This was announced in [PJ2], later revised and distributed in [PJ3, PJ4, PJ5].) (There have been further developments since then in the case $k = 2$; see the historical remarks below.) Much of the method we used then was peculiar to the case $k = 2$. We have developed now new estimates, more powerful and more general, with which we have extended the regularity theory to the dimensions in theorem A. This is the first general existence theorem of this type for regular k dimensional minimal surfaces when $k > 2$.

Generally speaking, there are two large parts to the logical development. In chapters 3 and 4 we derive one part, a very general existence theory for minimal surfaces, applicable on arbitrary compact riemannian manifolds in all dimensions and codimensions. Chapters 5, 6, and 7 compose the second part, a regularity theory, in which derive the special estimates necessary to establish the existence theorem A. The general existence theory has its roots in [AF1] and [AF2]. Almgren demonstrated [AF1] that on a compact manifold M , the homotopy groups of the integral cycle groups on M are isomorphic to the homology groups of M . This led him to construct a variational calculus in the large analogous to that of Marston Morse, from which he concluded [AF2] that M supports a nonzero stationary integral varifold in all dimensions not exceeding dim(M). Here we construct a similar variational calculus and prove what turns out to be a critical extension; namely, that M supports nonzero stationary integral varifolds with an additional variational property, which we have called <u>almost minimizing</u> (3.1). Intuitively one considers an almost minimizing varifold to be one which may be approximated arbitrarily closely by integral currents which are themselves very nearly locally area minimizing. Almost minimizing varifolds are principal objects of our investigation. The origins of the concept are quite natural (cf. 1.1 and 1.2). The main existence theorem is the following.

THEOREM B. GENERAL EXISTENCE THEOREM FOR MINIMAL SURFACES ON RIEMANNIAN MANIFOLDS (4.11). <u>For each</u> $k \leq n$, <u>every compact</u> n <u>dimensional riemannian</u>

manifold of class 4 supports a nonzero k dimensional stationary integral varifold which, at each point in the manifold, is almost minimizing in all small annular neighborhoods of that point.

The study of almost minimizing varifolds began in the first place because Almgren's theorem on the existence of stationary integral varifolds is inadequate to settle the question of existence of regular minimal surfaces on manifolds. This is because varifolds which are only stationary and integral have in general essential singularities, possibly of positive measure. If, in addition, the varifold is almost minimizing, then it possesses strong local stability properties which yield estimates on the singular sets. In particular, these estimates imply the singular set is empty for hypersurfaces of n dimensional manifolds, $3 \leq n \leq 6$, which is theorem A.

Thus, our regularity theory depends on careful analysis of stable surfaces (minimal surfaces whose second variation of area is nonnegative), a class of minimal surfaces which has been vigorously investigated in recent years (see [SJ], [LHB2], or [SSY], for example). For our purposes the salient property of stable surfaces is that their geometric configurations are considerably more restricted than those of general minimal surfaces. The principal descriptive result is this.

THEOREM C. DECOMPOSITION THEOREM FOR STABLE HYPERSURFACES (6.3). *If* $2 \leq k \leq 5$ *and* $M = \mathbb{R}^{k+1}$ (*or more generally,* M *is a* $(k+1)$-*dimensional submanifold of class* 5 *of* $\mathbb{R}^n$, *and* M *is sufficiently planar*), *then a stable* k *dimensional submanifold of* M *lying sufficiently near a cone is the disjoint union of minimal graphs of functions over a single k-plane.*

The proof of the decomposition theorem (as well as theorems D and E below) depends on a strong pointwise curvature estimate for stable surfaces due to Schoen, Simon, and Yau [SSY]. A derivation of this estimate based on [SSY] is in chapter 5.

We also prove an interesting compactness property of stable manifolds.

THEOREM D. COMPACTNESS THEOREM FOR REGULAR STABLE HYPERSURFACES (7.5). *If* $2 \leq k \leq 5$ *and* M *is a compact* $(k+1)$-*dimensional riemannian manifold of class* 5, *then the space of uniformly mass bounded, stable,* k *dimensional integral varifolds* *on M with regular support is compact in the weak topology.*

If $2 \leq k \leq 6$ and if C is a k dimensional cone in $\mathbb{R}^{k+1}$ which is stable and regular (except at the vertex), then it is well known that C must be a hyperplane. If $2 \leq k \leq 5$, then we generalize this as follows.

THEOREM E. DISK THEOREM FOR STABLE CONES (7.6). *If* $2 \leq k \leq 5$, *and if* C *is a* k *dimensional integral varifold in* $\mathbb{R}^{k+1}$ *which is a cone* (C need not have regular support), *then there is a sequence of stable* k *dimensional integral varifolds with regular support converging to* C *if and only if* C *is a hyperplane, possibly with multiplicity.*

Finally we obtain an interesting existence and regularity theorem analogous to the classic theorems of Morse-Tompkins [MT] and Shiffman [SM].

THEOREM F. EXISTENCE OF MINIMAL MANIFOLDS OF GENERAL CRITICAL TYPE (7.14). *If* $2 \leq k \leq 5$, C *is a* (k-1)-*dimensional integral cycle in* $\mathbb{R}^{k+1}$, T_1 *and* T_2 *are* k *dimensional integral currents,* $\partial T_1 = \partial T_2 = C$, *and* T_1 *and* T_2 *locally minimize area among all integral currents with boundary* C *, then there exists a* k *dimensional integral varifold* V *such that* V *is stationary with respect to deformations with compact support in* $\mathbf{R}^{k+1} \sim \operatorname{spt} C$,

$$\infty > \|V\|(\mathbf{R}^{k+1}) > \max\{\mathbf{M}(T_1), \mathbf{M}(T_2)\},$$

$\operatorname{spt} C \subset \operatorname{spt} \|V\|$, *and* $\operatorname{spt}\|V\| \sim \operatorname{spt} C$ *is a* k *dimensional, real analytic, minimal submanifold of* $\mathbf{R}^{k+1}$.

Now we summarize the history of the question of the existence of k dimensional minimal surfaces on a compact n dimensional manifold M . The classic case of $k = 1$ and n arbitrary was settled in 1951 by Lyusternik and Fet [LF], who showed that an arbitrary compact manifold always supports a closed geodesic, possibly with self-intersections. In 1929 Lusternik and Schnirelmann [LS] proved that if M is two dimensional and simply connected, then there exist on M at least three closed geodesics without self-intersections. Recently there has been considerable interest in showing that for arbitrary $n \geq 2$, M supports many closed geodesics.

There are many interesting results with various geometric and topological hypotheses on M ; the case $k = 1$ has become a subfield of its own. For an exhaustive discussion and bibliography, see [KW].

When $k = 2$, the first general theorem was the predecessor of theorem A in [PJ2], as described above. Since then there have been two major developments on the existence of minimal immersions of 2-manifolds. Sacks and Uhlenbeck [SU] have studied minimal immersions of spheres, the main result being that if $n \geq 3$ and if the universal covering space of M is not contractible, then there exists a smooth mapping of $\underline{\underline{S}}^2$ into M which (except perhaps at a finite number of branch points) is a conformal minimal immersion, possibly with self-intersections. When $n = 3$, there will be no branch points. In a second development, Schoen and Yau [SY] have proved that if S is a riemann surface, $f : S \to M$ is continuous, and the induced map of fundamental groups $f_{\#} : \pi_1(S) \to \pi_1(M)$ is injective, then there is a branched minimal immersion $g : S \to M$ such that $g_{\#} = f_{\#}$ and g minimizes induced area among all maps with the same action on $\pi_1(S)$. These authors have applied this result to analyze the topology of manifolds with nonnegative scalar curvature.

Other results include the following. We have already mentioned the theorem of Almgren [AF2] on the existence of stationary varifolds on arbitrary manifolds. Another important theorem is that if $k = n-1 \leq 6$ and the k dimensional homology group of M with coefficients in the integers does not vanish, then M supports a closed minimal hypersurface. This follows from the methods of [FH1, chapter 5] applied to a homologically area minimizing representative of a nonzero k dimensional homology class on M (cf. 7.2). Also, in [LHB1], Lawson explicitly constructs examples to show that $M = \underline{\underline{S}}^3$ supports closed two dimensional minimal submanifolds of arbitrarily high genus.

Insofar as possible, our presentation is self-contained. We have included, in particular, such techniques as we need from differential geometry and topology (4.6 excepted). Regarding geometric measure theory, we have not been so self-contained; Federer's exhaustive treatment [FH1] makes what would be a lengthy effort redundant at best. One important topic, varifolds, has appeared since the publication [FH1]; the best reference for this is the comprehensive monograph [AW1]. We have listed in

2.4 those theorems about varifolds which we need, so that it is not strictly necessary to have [AW1] in hand in order to follow our arguments.

We might say a few words about chapter 1. Although the complete development of the monograph is lengthy and not always easy, the fundamental ideas are natural and simple. Chapter 1 is an informal description of our methods, largely by illustrative examples. It is also a good source of examples and counterexamples for specific questions in the theory. We hope the reader finds it useful.

It is a pleasure to thank Professor F. J. Almgren, Jr., for helpful discussions. I am grateful to Mrs. Diane Strazzabosco for typing much of this manuscript.

REMARK ADDED IN PROOF. As described above, the curvature estimates of [SSY] for stable surfaces were essential in the regularity (theorem A), and also in theorems C, D, and E. The dimension restriction $2 \leq k \leq 5$ in these theorems reflected a corresponding restriction in [SSY]. Now R. Schoen and L. Simon have derived more general curvature estimates in a form applicable to stable k dimensional hypersurfaces for all positive integers k