Lectures on

General Relativity and Cosmology

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Lectures on
GENERAL RELATIVITY AND COSMOLOGY

To

My Father
Professor V. V. Narlikar
who has played a pioneering role
for over four decades
as a teacher and research guide
to the relativists in India

Preface

This book is based on the lecture course I give in the graduate school of the Tata Institute of Fundamental Research (TIFR), and is intended to be an introductory text for theoretical physicists. The reader is expected to be familiar with the standard techniques of mathematical physics, Newtonian mechanics and the special theory of relativity.

For many years since its inception, general relativity held the reputation of being a difficult subject, somewhat aloof from the rest of physics. Einstein himself was aware of this and it was his ambition to extend the radical concepts underlying the formulation of general relativity so as to include a unified description of the whole of physics. This goal remains remote even today.

Nevertheless, thanks to the observational inputs from cosmology and astrophysics, the gap between general relativity and the rest of physics has narrowed considerably. The theory has found applications to the large-scale structure of the universe and to the interiors of highly dense stars and supermassive objects. It has enriched physics with the concept of black holes. In these lectures I have emphasized the *physical* aspects of this theory rather than its mathematical ones, and the selection of the topics of the lectures has been dictated by this motive.

Scientific enquiry proceeds on the premise that no theory is perfect. General relativity, with all its underlying beauty, is not an exception to this rule. In the last three lectures in this book, I have discussed some of the shortcomings of general relativity and the attempts made by other theories to circumvent them. I have also discussed briefly the problems of quantization and the remarkable Hawking-effect that black holes radiate. In an introductory text of this kind such discussions have to be somewhat superficial. They are meant to convey to the reader the present state of excitement in these areas. A subject is exciting for research only when it is incomplete and full of difficulties.

No apology is made for the absence of certain topics like differential forms, tetrads, the Petrov classification, the techniques of global differential geometry, etc. While not denying their important role in the modern development of the subject, I strongly feel that they will be better appreciated and understood in a text written at a higher level than the present one.

In writing these notes I have preferred the lecture format because of its informality. The notional time of a lecture is assumed to be sixty minutes, although it would be rather ambitious to cover all the material of some of the longer lectures in that period. In practice I find it useful to leave certain gaps for the students to fill in. The problems given at the end of each lecture are also intended to supplement or amplify the material in the text. An extensive list of references given at the end of the book will be found useful by those wanting to obtain further details relating to matters referred to in the text. It is interesting to note that over two-thirds of the reference in this list relate to the work of the past two decades.

I was fortunate in having the benefit of advice from my father, Professor V. V. Narlikar, who read the first draft of most of these lecture notes. I thank Mr P. Joseph and Miss Margaret Abnes for typing the manuscript promptly. It is a pleasure to acknowledge the assistance that Mr Palekar and the staff of the TIFR drawing office provided for the illustrations in this book.

Bombay
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JAYANT V. NARLIKAR

Contents

	Preface	*vii*
LECTURE 1:	Theories of Gravitation	1
LECTURE 2:	Vectors and Tensors	10
LECTURE 3:	Tensor Calculus	27
LECTURE 4:	Space-Time Curvature	40
LECTURE 5:	Space-Time Symmetries	57
LECTURE 6:	Energy Momentum Tensors	72
LECTURE 7:	The Einstein Equations of Gravitation	90
LECTURE 8:	The Schwarzschild Solution	105
LECTURE 9:	The Experimental Tests of General Relativity	120
LECTURE 10:	Strong Gravitational Fields (I): Relativistic Astrophysics	141
LECTURE 11:	Strong Gravitational Fields (II): Black Holes	156
LECTURE 12:	Cosmology	173
LECTURE 13:	The Friedmann Models	188
LECTURE 14:	The Steady State Theory	204
LECTURE 15:	Observational Tests in Cosmology	216
LECTURE 16:	Mach's Principle	231
LECTURE 17:	Glimpses into Advanced Topics (I)	244
LECTURE 18:	Glimpses into Advanced Topics (II)	255
	References	271
	Index	275

LECTURE 1

Theories of Gravitation

Of all the basic intersections of physics, gravitation is the oldest known. In the late seventeenth century Isaac Newton formulated the law of gravitation. In his famous book *Principia* (1687), Newton[1]* discussed the laws of motion as well as of gravitation. With the help of these laws he was not only able to describe the behaviour of the falling apple—the phenomenon which is supposed to have inspired him to think about a law of gravitation—but also such diverse phenomena as the motion of projectiles on the Earth, the movement of planets round the Sun, the motion of the Moon round the Earth, and so on. The phenomenon of tides is also explained by the Newtonian law of gravitation. It was the implicit faith in this law that led Adams (1846) and LeVerrier (1846) to the discovery of the planet Neptune. Even today, this law forms the basis of the calculations of flight trajectories of spacecraft and satellites sent out from the Earth.

Yet, in spite of the all-embracing character of Newtonian gravitation and its string of successes, this law is considered unsatisfactory in the framework of modern theoretical physics. Why? It is necessary to answer this question before discussing more sophisticated theories of gravitation. Having answered this question we will then try to answer the more difficult question: what are the desirable features of a modern gravitation theory?

*The references are listed at the end of the book.

1.1 *The Conflict between Newtonian Gravitation and Special Relativity*

In 1905 Einstein[2] put forward the special theory of relativity. This theory revolutionized the concepts of space, time and motion on which the Newtonian laws were founded. The conflict between special relativity and Newtonian gravitation shows up in several different ways.

In Newtonian physics the three-dimensional space and the one dimensional time were kept apart. Using the Cartesian coordinates (x, y, z) to specify a point in space and t as the Newtonian time, the laws of Newtonian physics were covariant under the three-dimensional orthogonal group of transformations of space coordinates which preserve

$$r^2=(x^2+y^2+z^2), \tag{1.1}$$

apart from the group of Galilean transformations in space and time. Thus, the force of gravitational attraction between two masses m_1, m_2 separated by a distance r,

$$\mathbf{F}=-G\frac{m_1m_2}{r^3}\mathbf{r} \tag{1.2}$$

is covariant under this orthogonal group. The same applies to the Poisson equation analogue of Eq. (1.2):

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\equiv\nabla^2\phi=-4\pi G\rho \tag{1.3}$$

where ϕ is the gravitational potential of a distribution of matter with density ρ. In Eqs. (1.2) and (1.3), G is the constant of gravitation.

In special relativity, the orthogonal group above is replaced by the Lorentz group in three space plus one time dimensions. This preserves

$$s^2=c^2t^2-x^2-y^2-z^2, \tag{1.4}$$

where c is the speed of light. The Galilean transformations are replaced by the Lorentz transformations.

Neither Eq. (1.2) nor Eq. (1.3) are invariant under the Lorentz group. Physically this means that the speed of light has no invariant status in the Newtonian scheme. Indeed, the invariance of Eq. (1.1) and t separately in the Newtonian picture implies that there the gravitational influence propagates with infinite speed.

Faced with this conflict there is an apparent choice between two possible ways out. The first one is to retain the Newtonian framework and to argue against the special theory of relativity. This would also involve a modification of Maxwell's electromagnetic theory which is also Lorentz-covariant. The second approach is to abandon or modify Newtonian gravitation to make it compatible with the special theory of relativity.

The first alternative is ruled out by experiments. Motion of rapidly moving elementary particles shows that physical interactions, such as the electromagnetic interaction, are invariant under the Lorentz transformation, and not the Galilean transformation. Attempts to modify Maxwell's equations to make them invariant under Galilean transformations have not been successful.

The second alternative, although a more feasible one at first sight, is also not very productive. The reasons are as follows. A straightforward generalization of Eq. (1.3) to

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi\equiv\Box\phi=4\pi G\rho \tag{1.5}$$

does imply the propagation of the gravitational disturbance with the speed of light. However, it raises other questions. If ϕ is a scalar field, then ρ must also be scalar. On the other hand, the equivalence of matter and radiation under the special relativistis relation

$$E=mc^2 \tag{1.6}$$

requires us to recognize that any energy density along with the mass density should act as a source of gravitational potential. This means we should look upon ρ not as a scalar but as the time-time component of the second rank energy momentum tensor. This means ϕ itself is not a scalar but a component of a second rank tensor. Thus, Eq. (1.5) will have to be modified to

$$\Box\phi_{ik}=4\pi GT_{ik}. \tag{6}$$

What interpretation may be given to the tensor ϕ_{ik}? Already we are departing from the simple structure of Newtonian gravitation.

A second related difficulty of Eq. (1.5) is that it still does not answer the important question: how is light affected by gravitation?

One may try to answer this by postulating that a photon of frequency ν has a gravitational mass $h\nu/c^2$ where h is the Planck's constant. Such a solution has a somewhat ad hoc character.

A third difficulty is of a more serious nature, in that it shows up the incompatibility of the gravitational phenomenon with the concept of the inertial frame which is so basic to special relativity itself. Ideally an inertial frame is specified as one in which a particle with no force on it appears to move with a uniform velocity in a straight line. In practice how can we achieve the state of no force? It is possible to shield a piece of matter from other interactions but not from gravitation. The universality of gravitation means that any shielding mechanism must involve matter (and energy) and hence it must attract. The only way to visualize an inertial frame is to imagine it far away from any gravitating matter. A concept like this is clearly of no use to someone performing experiments on this Earth or to an astronomer whose observations relate to distant but massive gravitating objects.

1.2 *The General Theory of Relativity*

In formulating the general theory of relativity as a theory of gravitation, Einstein was aware of these points of conflict. In order to get round the last point, which seemed to demand a reexamination of the basic concepts of special relativity itself, he proposed a very bold and radical solution. This was based on the following type of reasoning. If gravitation has something of a permanent character, i.e., if it is an interaction which cannot be turned on and off at will, then it must be intrinsic to the region where it is located. Einstein identified this intrinsic property of a space-time region with its geometry.

To understand the nature of the identification

$$\textit{Gravitation} \equiv \textit{Space-Time Geometry} \tag{1.8}$$

we go back to the last point of Section 1.1. There the practica definition of the concept of inertial frames forces us to go far away from any gravitating matter. It is only in such a region of space-time that may we expect the special theory of relativity to apply accurately. As implied by Eq. (1.4) the geometry of space-time is pseudo-Euclidean* in special relativity. In the presence of gravitating matter the correct description of space-time geometry, according to Einstein, should be non-Euclidean. And, he sought to relate the intrinsic parameters of a non-Euclidean space-time geometry to the distribution of gravitating matter and energy. Thus, in the presence of such a source of gravitation the gravitational effects will not be described through an explicit external force but through the non-Euclidean nature of the space-time geometry.

It is best to illustrate this with an example. Suppose a projectile

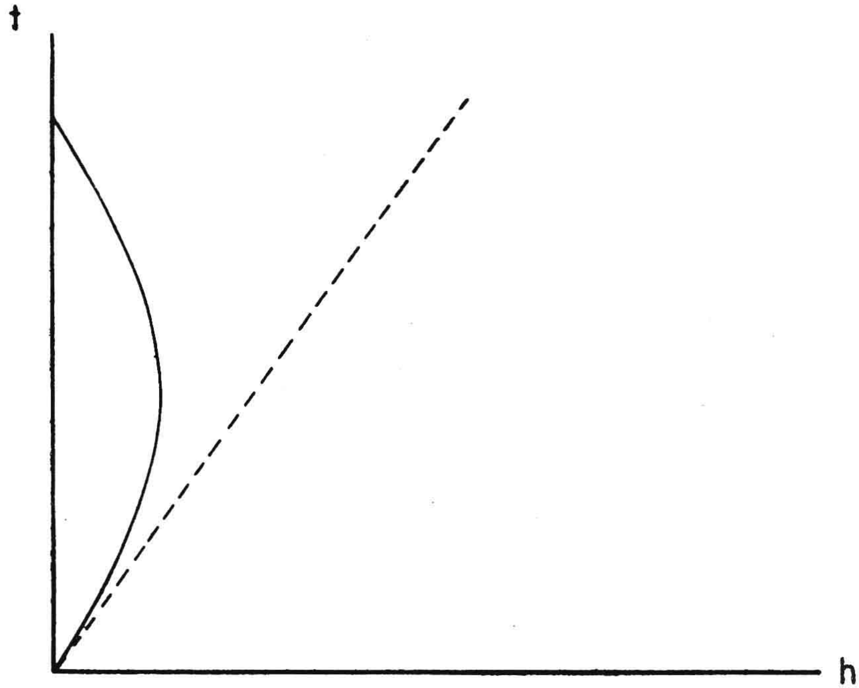

Fig. 1. The trajectory of a freely moving particle is shown by a dotted straight line. A particle moving under the Earth's gravity has the parabolic trajectory shown by the continuous line. In Newtonian gravitation, it is the Earth's gravitational force which bends the trajectory. In Einstein's view the continuous line is also 'straight' but the space-time is non-Euclidean because of Earth's gravity.

*In a Euclidean geometry the generalized Pythagoras theorem applies. In Eq. (1.4) the right-hand side has positive as well as negative squares. Hence the word 'pseudo'. Since this difference is not significant in many geometrical properties of space-time, we will drop this adjective where it causes no confusion. Thus the word 'non-Euclidean' will imply 'other than Euclidean as well as pseudo-Euclidean'.

is thrown vertically upwards with a velocity $\mathbf{v}$. If we assume for the moment that the Earth has no gravitational field and that the projectile is moving freely, we can deduce from Newton's first law of motion that it will continue to move in this direction with the same velocity $\mathbf{v}$ so that at a time t later, it will be at a height

$$h = |\mathbf{v}|\, t. \tag{1.9}$$

In the space-time diagram of Fig. 1 with h on the horizontal axis and t on the vertical axis, this trajectory is shown by a dotted straight line.

However, if we now take account of the downward acceleration due to the Earth's gravity of magnitude g, Eq. (1.9) is changed to

$$h = |\mathbf{v}|\, t - \tfrac{1}{2}gt^2 \tag{1.10}$$

and the track in the space-time diagram is a parabola. (In Fig. 1, this is shown by a continuous curve.)

According to Newton the deviation of this curve from the dotted straight line is due to the Earth's gravitational force. By contrast, according to Einstein, the second curve represents 'uniform motion in a straight line' but in a non-Euclidean space-time. Such an approach recognizes the permanency of Earth's gravitation and incorporates it in the geometry of the space-time.

Einstein[3] formulated the general theory of relativity in 1915, some ten years after the special theory. To what extent has this theory resolved the difficulties discussed in the last section? How good has it been on the observational front? These are the questions I shall return to, towards the end of this lecture course. I end this lecture with a brief discussion of another important concept which influenced Einstein in his formulation of general relativity.

1.3 *Mach's Principle*

When formulating his laws of motion Newton had been concerned with the notion of inertia and the inertial force. His second law of motion is written in the form

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{1.11}$$

where m is the mass and $\ddot{\mathbf{r}}$ the acceleration of a particle acted upon by an external force $\mathbf{F}$. The mass m, which is indicative of the resistance offered by the particle to a change of its velocity, is the measure of its inertia. The larger the mass of the particle, the less is the change in the velocity induced by a given force.

In the above equation an important question arises: what is the frame of reference? Suppose we denote it by S and then consider another frame S', accelerated with respect to S by $\mathbf{a}$. Suppose the external force is such that it does not change when measured in S'. Then we must have the following relation for the motion of the above particle in S':

$$m\ddot{\mathbf{r}}' = \mathbf{F} - m\mathbf{a} \tag{1.12}$$

where $\ddot{\mathbf{r}}'$ is its acceleration measured in S'. (It is assumed here that m, being the intrinsic property of the particle, does not change.) Thus, in S' the second law of motion applies provided we add an additional force $-m\mathbf{a}$. This force, which is proportional to the mass of the particle, is called the *inertial force*. An example of such a force is the so-called centrifugal force arising in a circular motion. An observer sitting on a stone tied to a string and whirled in a circle sees the stone at rest. In order to reconcile with the second law of motion he has to invent a centrifugal force to balance the pull of the taut string.

Clearly, the necessity of including inertial forces shows that not all frames of reference are equivalent, so far as the second law of motion is involved. There is a unique class of frames in which no inertial forces are required. What is the nature of a special frame in which no inertial force is needed? How is it identified observationally? Newton could do no better than postulate the existence of such a frame which he called the *absolute space*. Frames accelerated in relation to the absolute space would require suitable inertial forces. The Earth's rotation relative to the absolute space can be measured, for example, by the Foucault pendulum experiment which makes use of the inertial forces.

In the last century the philosopher and scientist Ernst Mach[4] criticized this ad hoc status of the absolute space. By then an astronomical result had emerged which showed how to identify the special frame of reference. It was the frame in which the distant stars are non-rotating.* In other words, Newton's absolute space

*Later observations indicate a slow rotation of the stars in our Galaxy. However, the external galaxies in the universe do show the non-rotating property to a high degree of accuracy.

can not only be identified by a local experiment but also by looking at the distant matter. For example, the Earth's rotation about its axis as measured by a Foucault pendulum experiment can be compared with that obtained relative to the distant stars. In either case we get the same answer. This coincidence, Mach argued, must have a deep significance which was not reflected in the formulation of Newtonian dynamics.

The fact that the local inertial frame is the one in which the distant stars are non-rotating, implied, according to Mach, an intimate connection between the background of distant matter and the concept of inertia. Remove the background and the concept of inertia becomes meaningless. Consider, for example, the idealized case of a single particle in an otherwise empty universe. Since it is not subject to any force, Eq. (1.11) becomes

$$m\ddot{\mathbf{r}} = \mathbf{0}. \tag{1.13}$$

What should we conclude from this? If we say $\ddot{\mathbf{r}} = \mathbf{0}$, then we get the particle moving with a uniform velocity—a meaningless result since there is no background to measure this velocity or to deduce its constancy. In the absence of a background frame of reference, $\ddot{\mathbf{r}}$ should have an indeterminate value. Such a conclusion is possible if $m=0$. The property of inertia therefore is not an intrinsic property of the particle but is dependent on the existence of a background.

This concept, which was not quantified by Mach, is called *Mach's principle.* It has had a mixed reception among the theoretical physicists. Some are unimpressed by the reasoning while others attach a lot of significance to it. Einstein himself took it seriously and hoped to incorporate it into general relativity. In this, as we shall see later, he was not very successful.

1.4 *The Desirable Features of a Gravitation Theory*

I now return to the question raised earlier: what should be the desirable features of a gravitation theory? This question can only be answered against the background of the current state of the rest of physics.

First, as discussed in Section 1.2, the theory should resolve the inconsistencies of blending Newtonian gravitation with the special theory of relativity. The fact that both these theories represent some approximation to the truth, suggests that a new, more sophisticated theory should resemble Newtonian gravitation when special relativistic effects are negligible and should reduce to special relativity in the approximation when the gravitational effects are small.

Other desirable features include a further understanding of Mach's principle, an easy adaptability with quantum physics and the suggestion (if not the proof) of a link of gravitation with other interactions of physics.

In these lectures which will deal mainly with general relativity, I shall discuss how far this theory possesses these features. I shall also describe briefly some of the alternative approaches to general relativity.

1.5 *Two Approaches to General Relativity*

A survey of literature on general relativity shows that the theory has been developed from two points of view: physical and mathematical. The physical aspect emphasizes that the primary aim of the theory is to explain the phenomenon of gravitation as observed in nature. The techniques used in this development are similar to those in the rest of theoretical physics. The space-time is described by four coordinates and all the physical and geometrical quantities are studied in terms of these coordinates. The other approach aims at looking at the intrinsic properties of space-time using, as far as possible, coordinate-free techniques.

There is something to be said for either approach. The physical approach, which will be adopted in these lectures, is useful in studying gravitation in relation to the rest of physics. It is certainly needed in solving specific problems of gravitating objects. However, when looking at certain global properties of space-time this approach is at a disadvantage compared to the intrinsic method. The latter is able to yield powerful theorems on space-time structure, e.g., singularities, horizons, trapped surfaces, etc., which cannot be so elegantly described in a coordinate-dependent description. Nevertheless, an understanding of the coordinate-dependent physical approach is necessary in order to appreciate the power of the intrinsic one. Hence a first course on general relativity should emphasize the coordinate-dependent physical approach.