LINEAR PROGRAMMING

METHODS AND APPLICATIONS

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PREFACE

The material in this book was originally prepared for an introductory course in linear programming given at the Graduate School, U.S. Department of Agriculture, Washington, D.C. In developing and expanding the notes into a suitable text, I have attempted to pursue the same objectives that guided the presentation of the course material. These basic aims were to instill in the student an ability to recognize potential linear-programming problems, to formulate such problems as linear-programming models, to employ the proper computational techniques to solve these problems, and to understand the mathematical aspects that tie together these elements of linear programming.

It is very convenient to divide the subject matter of linear programming into three separate, but not distinct, areas: *theoretical*, *computational*, and *applied*. In teaching the course, I found it appropriate, instructive, and beneficial to the students to interlace material from all three areas as much as possible. Hence, after an introductory lecture on applications and the mathematical model of linear programming (Chap. 1), the mathematics of convex sets and linear inequalities were developed and followed by the computational aspects of the elimination method for solving linear equations (Chap. 2). The mathematical properties of a solution to the general linear-programming problem were next evolved. Then, in an attempt to explain fully the fundamentals of the simplex computational procedure, the ability to generate extreme-point solutions was shown to be a simple variation of the elimination technique of Jordan and Gauss (Chap. 3). The next set of lectures developed the theoretical and computational elements of the simplex method of G. B. Dantzig (Chap. 4). A discussion on the duality problems of linear programming (Chap. 5) was followed by lectures on the formulation of certain illustrative applications (Chaps. 10 and 11). The final lectures of the course[1] described the relationship between linear programming and the zero-sum two-person game (Chap. 12).

The basic-course notes have been revised to include full discussion of the revised simplex method (Chap. 6), degeneracy procedures (Chap. 7), parametric programming (Chap. 8), further computational techniques

[1] This one-semester course consisted of 16 evening lectures of 2½ hours' duration.

(Chap. 9), and other topics and applications. As a result of a suggestion to the author, all the material has been gathered into three parts: an introduction, methods, both theoretical and computational, and applications. It is felt that this arrangement enhances the usefulness of the book for reference, and presents the three areas of linear programming in a related but separate manner. Consequently, the reader will find some relatively advanced topics, such as the revised simplex method and parametric linear programming, appearing before the basic discussions of the transportation problem and general applications. It is suggested that, in order to motivate one's study of linear programming, the chapters should not be studied in numerical sequence. Instead, one should, as soon as possible (probably after Chap. 4 or 5), become acquainted with material in the applications sections.[1]

It is felt that the material covered in this text is appropriate for use in mathematics courses at the senior or first-year-graduate level. However, because of the interest in linear-programming methods outside the academic field, it seemed advisable to include sufficient material on matrices and vectors to make the work complete for all readers (Chap. 2). It should be noted that much of the mathematical notation used in subsequent chapters is developed in Chap. 2.[2]

The search for the best, the maximum, the minimum, or, in general, the optimum solutions to a variety of problems has entertained and intrigued man throughout the ages. Euclid, in Book III, was concerned with finding the greatest and least straight lines that can be drawn from a point to the circumference of a circle, and in Book IV he described how to find the parallelogram of maximum area with a given perimeter. However, the rigorous approach to these and more sophisticated problems had to wait until the great mathematicians of the seventeenth and eighteenth centuries developed the powerful methods of the calculus and the calculus of variations. With these techniques we can find the maximum and minimum solutions to a wide range of optimization problems. These and other mathematical optimization procedures were mainly concerned with the solutions to problems of a geometric, dynamic, or physical nature. Such problems as finding the minimum curves of revolution and the curve of quickest descent are resolved by these classical optimization methods.

Recently, a new class of optimization problems has originated out of the complex organizational structures that permeate modern society.

[1] In a course that meets three times a week, one lecture could be devoted to applications and/or reports of case studies cited in the Bibliography.

[2] The student will soon find that one of the main difficulties in understanding the mathematics of linear programming arises from the diverse and often intricate notation used in many of the source papers of this field. Wherever possible, I have employed "standard," consistent, and, I hope, explicit notation.

Here we are concerned with such matters as the most efficient manner in which to run an economy, or the optimum deployment of aircraft that maximizes a country's chances of winning a war, or with such mundane tasks as mixing the ingredients of a fertilizer to meet agricultural specifications at a minimum cost. Research on how to formulate and solve such problems has led to the development of new and important optimization techniques. Among these we find the subject of this book—*linear programming*. The linear-programming model, i.e., the optimization of a linear function subject to linear constraints, is simple in its mathematical structure but powerful in its adaptability to a wide range of applications.

Historically, the general problem of linear programming was first developed and applied in 1947 by George B. Dantzig, Marshall Wood, and their associates of the U.S. Department of the Air Force. At that time, this group was called on to investigate the feasibility of applying mathematical and related techniques to military programming and planning problems. This inquiry led Dantzig to propose "that interrelations between activities of a large organization be viewed as a linear programming type model and the optimizing program determined by minimizing a linear objective function." In order to develop and extend these ideas further, the Air Force organized a research group under the title of Project SCOOP (Scientific Computation of Optimum Programs). Besides putting the Air Force programming and budgeting problems on a more scientific basis, Project SCOOP's major contribution was the formal development and application of the linear-programming model. These early applications of linear-programming methods fell into three major categories: military applications generated by Project SCOOP, interindustry economics based on the Leontief input-output model, and problems involving the relationship between zero-sum two-person games and linear programming. In the past 10 years these areas of applications have been extended and developed, but the main emphasis in linear-programming applications has shifted to the general industrial area.

The initial mathematical statement of the general problem of linear programming was made by Dantzig in 1947 along with the *simplex method*, a systematic procedure for solving the problem. Prior to this a number of problems (some unsolved) were recognized as being of the type that dealt with the optimization of a linear function subject to linear constraints. The more important examples include the transportation problem posed by Hitchcock (1941) and independently by Koopmans (1947) and the diet problem of Stigler (1945). The first successful solution of a linear-programming problem on a high-speed electronic computer occurred in January, 1952, on the National Bureau of Standards SEAC machine. Since that time, the simplex algorithm, or variations of this procedure, has been coded for most of the intermediate and large general-purpose electronic computers in the United States and England.

Linear programming has become an important tool of modern theoretical and applied mathematics. This remarkable growth can be traced to the pioneering efforts of many individuals and research organizations. Specifically, I would like to make special mention of George B. Dantzig, Murray A. Geisler, Leon Goldstein, Julian L. Holley, Walter W. Jacobs, Alex Orden, Emil D. Schell, and Marshall K. Wood, all formerly with the U.S. Department of the Air Force; Leon Gainen, Alan J. Hoffman, and Solomon Pollack, formerly with the National Bureau of Standards; and the research groups of the Graduate School of Industrial Administration of the Carnegie Institute of Technology, The RAND Corporation, the Department of Mathematics of Princeton University, and the Cowles Commission for Research in Economics. I would like to thank the above named individuals and groups, other authors, and their publishers for their kind permission to use certain basic material contained in what might be considered "source documents" of linear programming. Appropriate references are given in the text.

This edition includes new sections on sensitivity analysis, integer programming, and the decomposition algorithm. The latter two subjects represent important recent advances to the computational aspects of linear programming. Additional material and exercises have been included in a number of sections. The section on Available Digital-computer Codes has been updated, and many new publications have been included in the references.

I wish to acknowledge the initial encouragement to write this text by my former associates at the Directorate of Management Analysis of the Department of the Air Force and to thank Harold Fassberg, Walter W. Jacobs, Thomas L. Saaty, and Kenneth Webb for their many valuable suggestions. Special appreciation is due Mrs. Thelma Chesley and Mrs. Anne Bache for their excellent typing of the original and second-edition manuscripts, respectively.

Saul I. Gass

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part 1

INTRODUCTION

chapter 1

GENERAL DISCUSSION

1. Linear-programming Problems

Programming problems are concerned with the efficient use or allocation of limited resources to meet desired objectives. These problems are characterized by the large number of solutions that satisfy the basic conditions of each problem. The selection of a particular solution as the best solution to a problem depends on some aim or over-all objective that is implied in the statement of the problem. A solution that satisfies both the conditions of the problem and the given objective is termed an *optimum solution*. A typical example is that of the manufacturer who must determine what combination of his available resources will enable him to manufacture his products in a way which not only satisfies his production schedule, but also maximizes his profit. This problem has as its basic conditions the limitations of the available resources and the requirements of the production schedule, and as its objective the desire of the manufacturer to maximize his gain.

We shall consider only a very special subclass of programming problems called *linear-programming problems*. A linear-programming problem differs from the general variety in that a *mathematical model* or description of the problem can be stated, using relationships which are called "straight-line," or linear. Mathematically, these relationships are of the form

$$a_1x_1 + a_2x_2 + \cdots + a_jx_j + \cdots + a_nx_n = a_0 \dagger$$

where the a_j's are known coefficients and the x_j's are unknown variables. The complete mathematical statement of a linear-programming problem includes a set of simultaneous linear equations which represent the conditions of the problem and a linear function which expresses the objective of the problem. In Sec. 2 we shall state a number of programming problems and formulate them as linear-programming problems.

To solve a linear-programming problem, we must initially concern ourselves with the solution of the associated set of linear equations. There are various criteria which can be applied to a set of linear equations

† Geometrically, these relationships are equivalent to straight lines in two dimensions, planes in three dimensions, and hyperplanes in higher dimensions.

to reveal whether a solution or solutions to the problem exist (see Dickson [34]).[1] The set of two equations in two variables

$$\begin{aligned} 2x_1 + 3x_2 &= 8 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

has the *unique* solution $x_1 = 1$ and $x_2 = 2$, while the single equation

$$x_1 + 2x_2 = 8 \tag{1.1}$$

has an *infinite* number of solutions. From (1.1) we have

$$x_1 = 8 - 2x_2 \quad \text{or} \quad x_2 = 4 - \tfrac{1}{2}x_1$$

For every value of x_2 (or x_1) there is a corresponding value of x_1 (or x_2). If we further restrict the variables to be nonnegative, i.e., $x_1 \geq 0$ and $x_2 \geq 0$, we limit the range of the variables, since

$$x_1 = 8 - 2x_2 \geq 0 \quad \text{implies} \quad 0 \leq x_2 \leq 4$$

and

$$x_2 = 4 - \tfrac{1}{2}x_1 \geq 0 \quad \text{implies} \quad 0 \leq x_1 \leq 8$$

We still have an infinite number of solutions, but the addition of further restrictions or constraints to (1.1) has resulted in less freedom of action. As we shall show, the condition of nonnegativity of the variables is an important requirement of linear-programming problems. Systems like (1.1) in which there are more variables than equations are called underdetermined. In general, underdetermined systems of linear equations have either no solution or an infinite number of solutions.

One important method of determining solutions to underdetermined systems of equations is to reduce the system to a set containing just as many variables as equations, i.e., a determined set. This can be accomplished by letting the appropriate number of variables equal zero. For example, the underdetermined system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 8 \\ x_1 + 2x_2 + 2x_3 &= 5 \end{aligned} \tag{1.2}$$

has three such solutions:

$$\begin{aligned} &x_1 = 0 \qquad &x_2 = 1\tfrac{1}{4} \qquad &x_3 = -\tfrac{1}{4} \\ &x_1 = 1\tfrac{1}{3} \qquad &x_2 = 0 \qquad &x_3 = \tfrac{2}{3} \end{aligned}$$

[1] Numbers in brackets refer to the publications listed in the References at the back of the book.

and

$$x_1 = 1 \qquad x_2 = 2 \qquad x_3 = 0†$$

Mathematically, linear programming deals with *nonnegative solutions* to underdetermined systems of linear equations. As we shall show in the succeeding chapters, the only solutions we have to be concerned with are those corresponding to determined subsets of equations that have been obtained in the manner described above. If, for example, we let Eqs. (1.2) represent the conditions of a linear-programming problem, we need only to consider the two nonnegative solutions $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $x_1 = 1\frac{1}{3}$, $x_2 = 0$, $x_3 = \frac{2}{3}$. The remaining solutions fail to satisfy either the nonnegativity requirements or other criteria to be discussed.

Just as the general programming problem has some objective that guides the selection of the solution to be used, the linear-programming problem has a linear function of the variables to aid in choosing a solution to the problem. This linear combination of the variables, called the *objective function*, must be optimized by the selected solution. If for (1.2) we wished to maximize the objective function $x_1 + x_2 + x_3$, then, of the two nonnegative solutions, the solution $x_1 = 1\frac{1}{3}$, $x_2 = 0$, $x_3 = \frac{2}{3}$ is the optimum, as it yields a value of $1\frac{3}{3}$ for the objective function compared to a value of 3 for the other nonnegative solution. If we wanted to minimize the objective function $x_1 - x_2$, then the solution $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ would be the optimum, with a value of -1. As we have implied, the optimum solution either maximizes or minimizes some linear combination of the variables. Since the maximum of a linear function is equal to minus the minimum of the negative of the linear function, we lose no generality by considering only the minimization problem.

With the added condition of optimizing an objective function, we are now able to select a single solution that satisfies all the conditions of the problem. There might be multiple solutions in that more than one nonnegative solution to the equations gives the same optimum value of the objective function. Generally speaking, combining the linear constraints of the programming problem with the optimization of a linear objective function transforms an underdetermined system of linear equations that describes a programming problem with many possible solutions to a system that can be solved for a solution that yields the unique optimum value of the objective function.

We next give the general mathematical statement of the linear-programming problem:

† There are, of course, an infinite number of other solutions to (1.2), which can be obtained by arbitrarily setting one of the variables equal to a constant; e.g., with $x_1 = a$ we have $x_2 = (11 - 3a)/4$ and $x_3 = (-1 + a)/4$.

Minimize the objective function

$$c_1x_1 + c_2x_2 + \cdots + c_jx_j + \cdots + c_nx_n$$

subject to the conditions

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ &\vdots \\ x_j &\geq 0 \\ &\vdots \\ x_n &\geq 0 \end{aligned}$$

and

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n &= b_i \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the c_j for $j = 1, 2, \ldots, n$; b_i for $i = 1, 2, \ldots, m$; and a_{ij} are all constants, and $m < n$. The c_j are called *cost coefficients.*

As will be discussed in the following chapters, every linear-programming problem has either:

1. No solution in terms of nonnegative values of the variables
2. A nonnegative solution that yields an infinite value to the objective function
3. A nonnegative solution that yields a finite value to the objective function

A linear-programming problem that describes a valid, practical programming problem usually has a nonnegative solution with a corresponding finite value of the objective function.

2. Examples of Linear-programming Problems

To illustrate the application of the above mathematical description of the linear-programming model, we shall next discuss the linear-programming formulation of three problems. A more detailed discussion of these and other problems is given in Part 3.

The Transportation Problem. A manufacturer wishes to ship a number of units of an item from several warehouses to a number of retail stores. Each store requires a certain number of units of the item, while each warehouse can supply up to a certain amount. Let us define the following:

m = the number of warehouses
n = the number of stores
a_i = the total amount of the item available for shipment at warehouse i
b_j = the total requirement of the item by store j
x_{ij} = the amount of the item shipped from warehouse i to store j

We shall assume that the total amount available is equal to the total required, that is, $\sum_i a_i = \sum_j b_j$. As will be shown later, this assumption is not a restrictive one.

The x_{ij} are the unknown shipments to be determined. If we form the array (for $m = 2$ and $n = 3$)

		Stores 1	Stores 2	Stores 3	
Warehouses	1	x_{11}	x_{12}	x_{13}	a_1
	2	x_{21}	x_{22}	x_{23}	a_2
		b_1	b_2	b_3	

we see that the total amount shipped from warehouse 1 can be expressed by the linear equation

$$x_{11} + x_{12} + x_{13} = a_1 \tag{2.1}$$

For warehouse 2, we have

$$x_{21} + x_{22} + x_{23} = a_2 \tag{2.2}$$

We also note that the total amounts shipped to the three stores are expressed by the equations

$$\begin{aligned} x_{11} + x_{21} &= b_1 \\ x_{12} + x_{22} &= b_2 \\ x_{13} + x_{23} &= b_3 \end{aligned} \tag{2.3}$$

The manufacturer knows the cost c_{ij} of shipping one unit of the item from warehouse i to store j. We have the additional assumption that the cost relationship is linear; i.e., the cost of shipping x_{ij} units is $c_{ij}x_{ij}$.

The manufacturer wishes to determine how many units should be sent from each warehouse to each store so that the total shipping cost is a minimum. This objective of minimizing the cost is achieved by minimizing the linear cost function

$$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} \tag{2.4}$$

Since a negative x_{ij} would represent a shipment from store j to warehouse i, we require that all the variables $x_{ij} \geq 0$.

By combining Eqs. (2.1) to (2.3), the objective function (2.4), and the condition of nonnegativity of the variables, the transportation problem for $m = 2$ and $n = 3$ can be formulated in terms of the following linear-programming problem:

Minimize the cost function

$$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23}$$

subject to the conditions

$$\begin{array}{llllllll} x_{11} & & & & & & \geq 0 \\ & x_{12} & & & & & \geq 0 \\ & & x_{13} & & & & \geq 0 \\ & & & x_{21} & & & \geq 0 \\ & & & & x_{22} & & \geq 0 \\ & & & & & x_{23} & \geq 0 \end{array}$$

and

$$\begin{array}{lllllll} x_{11} + x_{12} + x_{13} & & & & = a_1 \\ & x_{21} + x_{22} + x_{23} & & & = a_2 \\ x_{11} & + x_{21} & & & = b_1 \\ x_{12} & + x_{22} & & & = b_2 \\ x_{13} & + x_{23} & & & = b_3 \end{array}$$

Activity-analysis Problem. A manufacturer has at his disposal fixed amounts of a number of different resources. These resources, such as raw material, labor, and equipment, can be combined to produce any one of several different commodities or combinations of commodities. The manufacturer knows how much of resource i it takes to produce one unit of commodity j. He also knows how much profit he makes for each unit of commodity j produced. The manufacturer desires to produce that combination of commodities which will maximize the total profit. For this problem, we define the following:

m = the number of resources
n = the number of commodities
a_{ij} = the number of units of resource i required to produce one unit of commodity j
b_i = the maximum number of units of resource i available
c_j = profit per unit of commodity j produced
x_j = the level of activity (the amount produced) of the jth commodity

The a_{ij} are sometimes called input-output coefficients.

The total amount of the ith resource that is used is given by the linear expression