Electromagnetic Wave Propagation in Turbulence

Evaluation and Application of Mellin Transforms

SECOND EDITION

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SECOND EDITION

To my wife Joan whose love and support has meant so much

Preface to the first edition

This book is directed at two audiences: those interested in problems of electromagnetic wave propagation in turbulence and those interested in evaluating integrals. For the first group, the text provides a systematic way to obtain analytic answers to problems in which the scintillation is small and there are no nonlinear effects due to high optical powers. For those interested in evaluating integrals, the integration method is explained in separate chapters. In the chapters containing examples of wave propagation in turbulence, the problem is quickly reduced to one of evaluating an integral, and can be viewed as examples of the integration technique.

To address these two audiences, this book develops a systematic way of expressing solutions to problems of electromagnetic wave propagation in turbulence in integral form. It also develops Mellin transform techniques that are used to evaluate these integrals. This technique has three major advantages over others: 1) it is applicable to a wide range of problems; 2) the application of the technique is straightforward; and 3) the answers are expressed in analytic form. Mellin transform and hypergeometric functions have been a scientific backwater and are used regularly by only a few people. That is a shame for several reasons. Mellin transforms allow a deeper understanding of infinite series. Knowing the Mellin transform of a function is tantamount to knowing its infinite series. Mellin transform techniques, which require an understanding of hypergeometric functions, enable one to deepen his or her understanding of elementary and transcendental functions. In addition to this pedantic usefulness, it is a natural way to solve several types of problems that have wide applicability. For instance, Mellin transforms permit one to perform integrations that are very difficult to perform by other means. They enable one to solve boundary value problems in spherical and cylindrical coordinates with the same ease that Fourier transform techniques afford in solving differential equations with constant coefficients. The self-similar characteristic of Mellin transforms leads to applications in image, radar and acoustic processing, and chaos and fractal theory.

The major part of this book develops and applies a method for evaluating integrals analytically and expressing the result either as infinite series or as a sum of generalized hypergeometric functions. At first look, the method to evaluate integrals is formidable, and the final results look very complicated. It has been suggested that results from a numerical integration can be obtained more quickly. One has to overcome these prejudices. It is true that the formalism is difficult to learn because it uses mathematical techniques that are generally unfamiliar to most scientists, but that was not a valid excuse for not learning other difficult techniques, which are part of a standard scientific education. If the formalism produces results that are useful, one should be willing to overcome the initial difficulty.

This technique has indeed proved to be very useful! The expressions for the power series, although lengthy to write down, are easily and quickly evaluated using the recursion relation for gamma functions. Recently several computer algebra programs have acquired the ability to evaluate generalized hypergeometric functions, to which the power series are equivalent. In this context the results are no more difficult to evaluate and plot for specific cases than results expressed in terms of more familiar functions such as sinusoids and exponentials. The analytic form of the answer uncovers the natural parameters of a problem and gives one insight into how important a parameter is — an insight that is difficult to develop with numerical techniques. Integrands that contain the difference of two almost equal quantities, a condition that leads to difficulties in numerical integration, are handled in the complex plane by simply deforming an integration path past a pole. Because the technique is algorithmically based, one can develop a computer algebra program that automatically evaluates these integrals, in which case the user would not need to learn the details of the technique to get an analytic answer and to generate curves for specific ranges of parameter values. The development of such a program is being investigated. Just as scientific calculators made tables of trigonometric functions and logarithms obsolete, such a computer program would do the same to most material in integral tables. It would also allow one to evaluate many integrals that are not in the integral tables.

This technique was originally developed to evaluate integrals one encounters when solving problems of electromagnetic wave propagation in turbulence. The technique enables one to solve problems in terms of integrals that are generated with filter functions that multiply the turbulence spectrum. Problems that take days to solve when one starts from first principles can often be solved in less than an hour with appropriate filter functions and Mellin transform techniques.

The techniques given in this book were developed over several years in the high-energy beam-control and propagation group at MIT Lincoln Laboratory. I would like to thank MIT Lincoln Laboratory for providing the opportunity to work on challenging problems for which this technique was developed, for the freedom to pursue research in this area, and for the chance to interact with people interested in helping to develop this technique. This work was sponsored by the Strategic Defense Initiative through the Department of the Air Force under Contract No. F19628-90-C-0002.

I chose to solve many of the problems to illustrate the method developed in this book. I did not do a comprehensive literature search to see if these problems were previously solved. I apologize if I have left out relevant references.

Several people provided ideas that enabled me to develop the technique. Lee Bradley first suggested the possible usefulness of Mellin transform techniques and the existence of the Marichev text. He also suggested the use of Gegenbauer polynomials in addressing the anisoplanatism problem. The technique of eval-

uating integrals in several complex planes was developed in collaboration with John Shelton.

Many people had a hand in suggesting what material to include, and how it should be organized. I would like to especially thank Jonathan Shonfeld for carefully reading the first draft and suggesting many organizational changes. I have had many suggestions from Robert Kramer, Hernan Praddaude, Ronald Parenti, and Charles Primmerman. Fred Knight was very helpful in converting the text into LaTeX. Jim Eggert was particularly helpful since he was willing to read the text during several stages of the evolution of this book and made many useful suggestions. The series editor Professor Hermann Haus and the Springer-Verlag editor Helmut Lotsch made helpful suggestions on how the material should be presented. I want to thank Bill Breen, Ed Sullivan, Dave Tuells, Kevin Walsh, and their staff for producing the figures, converting them into Postscript, and printing the final copy. Sue Richardson and Katharine Krozel provided useful editorial help.

A book like this that contains so much new material and has so many complicated equations is very difficult to make error free. I would appreciate hearing any comments you have on the material or errors you have found in the text. My E-mail address is Sasiela@ll.mit.edu.

Writing a book takes a tremendous investment in time and energy that is no longer available for home life. I thank my wife Joan for being so understanding during this period.

October, 1993 Richard Sasiela
Lexington, Massachusetts

Comments about the second edition:

The first edition was published by Springer-Verlag. This edition corrects typographical errors in that edition. The treatment of tilt of uncollimated beams was incorrect in Sections 4.5, and 4.6 because a γ that should have multiplied the diameter was missing. It was pointed out by Jan Herrmann that it was necessary to use the local tilt in these sections.

As pointed out by Byron Zollars, there were some internal inconsistencies with 2π factors in the development of the general formula for variance due to turbulence. This affected some intermediate formulas in Chapters 2 and 3.

Since the propagation of focused beams has become more important, this case has been treated more carefully and extensively.

The derivation of the basic equations for variance and the removal of Zernike terms is developed more carefully.

Many problems can be solved by using the filter functions for variance. For more complicated problems one needs to start with the filter functions for phase or log-amplitude and develop the variance filter functions from these. Several examples on how to do this are illustrated.

Computer algebra programs have become more powerful and many of the integrals can now be solved with these programs. Solving these problems by hand is time consuming and error prone. Having these programs to do this part of the analysis is very helpful.

Typically, one is interested in the Strehl ratio. Analytic solutions are obtained for the variances. The approximations for the Strehl ratio using the phase variance do not give accurate results for many cases of interest. The use of filter functions in the structure function is elaborated in this edition. The problem of finding the Strehl ratio when the structure function is a function of aperture position is addressed. Examples of solving for the Strehl ratio numerically are given.

Chapter 6 of the original book discussed other uses for Mellin transforms. This chapter was not needed for the development of the subsequent chapters. Since I have nothing new to add on this subject, the chapter was eliminated because of the additional topics that were addressed.

I want to thank Ronald Parenti who I have worked with on turbulence problems for over 30 years. Our recent collaboration with Professors Larry Andrews and Ronald Philips has been very productive.

Recent computer code results indicate that the calculation of the scintillation for finite beams based on Rytov theory is in error. The beam wave theory predicts a dip in the scintillation index for Fresnel number around unity. Code results predict a smaller dip. Apparently, the perturbation theory that starts with a diffraction-limited beam on axis is incorrect. In the region of error the tilt can be comparable to the beam diameter. In addition, the focus term caused by turbulence causes a change in beam size, which violates the diffraction-limited assumption. Various authors have corrected the Rytov scintillation by separately including the effects of jitter and beam spreading.

I want to thank Seth Trotz for solving the many problems encountered in converting this document into LaTeX 2_ε. Also, I want to thank Eric P. Magee and his students for pointing out errors in the draft copy of this edition.

Beth Huetter of SPIE helped to correct errors and produce a uniform format.

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February, 2007 Richard Sasiela
Lexington, Massachusetts

Glossary

$(a)_k$	Pochhammer symbol
$A(\boldsymbol{\kappa}, z)$	Amplitude of second wave
$B_\phi(d)$	Phase correlation function
c	normalized wavenumber, speed of light
C_n^2	Turbulence strength
$C_p^\lambda(x)$	Gegenbauer polynomial
D	Aperture diameter
D_i	Inner diameter of annulus
D_s	Diameter of a finite source
$\mathcal{D}(\mathbf{d}, \boldsymbol{\rho})$	Structure function of total turbulence
D_c	Characteristic source diameter for scintillation averaging
$\mathcal{D}_v(\mathbf{r})$	Structure function of velocity
$\mathcal{D}_n(\mathbf{r})$	Structure function of atmospheric density
$\mathcal{D}_\phi(\mathbf{d}, \boldsymbol{\rho})$	Structure function of phase
$\mathcal{D}_\chi(\mathbf{d}, \boldsymbol{\rho})$	Structure function of log-amplitude
$E_n(\boldsymbol{\kappa}, z)$	Normalized axial correlation function of turbulence
$f(\kappa)$	Normalized turbulence spectrum
$F(\Omega_1, \ldots, \Omega_k)$	$(n_1!)^{\Omega_1} \cdots (n_k!)^{\Omega_k}$
$F(\gamma\boldsymbol{\kappa})$	Aperture filter function
F_N	Fresnel number
${}_pF_q[(a);(b);z]$	Generalized hypergeometric function
$g(k)$	Hill Spectra
$g(\boldsymbol{\rho})$	Aperture weighting function
$G(\gamma\,\boldsymbol{\kappa})$	Complex aperture filter function
$G_{p,q}^{m,n}\left[z \middle\vert \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right]$	Meijer's G-function
H	End altitude of propagation path
$H_{p,q}^{m,n}\left[z \middle\vert \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right]$	Fox's H-function
$J_\nu(x)$	Bessel function
$\mathbf{k}$	Three-dimensional propagation vector $(\boldsymbol{\kappa}, k_z)$

k_0	$2\pi/\lambda$, wavenumber
k_z	Propagation vector along propagation direction
$K(\alpha)$	Modulation transfer function of a circular aperture
$K_\nu(x)$	Modified Bessel function of the third kind
L	End point of propagation path
L_i	Inner scale with definition $L_i = 2\pi/\kappa_i$
L_{in}	Inner scale with definition $L_{in} = 5.92/\kappa_i$
L_o	Outer scale
$n(\mathbf{r})$	Variation of air density with position
$P(\gamma, \kappa, z)$	Diffraction parameter
r_o	Coherence diameter of a plane wave
r_{os}	Coherence diameter of a spherical wave
R_o	Radius of curvature of beam wave at the source
$s,\ s_n,\ t$	Complex variables
$S(p_1, \ldots, p_k)$	$\sum_{n_1=p_1}^{\infty} \cdots \sum_{n_k=p_k}^{\infty} \frac{(-1)^{n_1+\cdots+n_k}}{n_1!\cdots n_k!}$
$S_\phi(\omega)$	Power spectral density of phase
$S_\chi(\omega)$	Power spectral density of log-amplitude
SR	Strehl ratio
t	Time or second complex variable
$\mathbf{u}_z$	Unit vector in propagation direction
$U(x)$	Heaviside unit step function
v_n	n^{th} velocity moment of turbulence
v_g	Wind ground speed
$v(h)$	Wind speed as a function of height
W_o	$1/e^2$ radius of beam wave
$Z(m,n)$	Zernike polynomial
$\alpha(h)$	Normalized atmospheric density versus altitude
γ	Propagation parameter
$\Gamma[x]$	Gamma function
$\delta(x-a)$	Dirac delta function
Δn_0	Difference of refractive index between two colors
θ_o	Isoplanatic angle
θ_χ	Characteristic source angle for scintillation reduction
θ_s	Angle subtended by a finite source
κ	Transverse wavenumber
κ_i	Wavenumber of inner scale
κ_o	Wavenumber of outer scale

λ	Wavelength
μ_m	m^{th} turbulence moment
μ_m^+	m^{th} upper turbulence moment
μ_m^-	m^{th} lower turbulence moment
$\nu(\boldsymbol{\kappa}, z)$	Fourier transform of refractive index variations
$\boldsymbol{\rho}$	Two-dimensional transverse spatial vector
τ	Time delay between measurement and correction
$\varphi(\boldsymbol{\rho}, z)$	Phase variation due to turbulence
$\varphi R(\boldsymbol{\rho}, z)$	Phase related quantity
$\Phi_1(\boldsymbol{\rho}, z)$	Total variation of the turbulence fluctuations
ψ	Psi or digamma function
$\chi(\boldsymbol{\rho}, z)$	Log amplitude due to turbulence
$\chi R(\boldsymbol{\rho}, z)$	Log-amplitude related quantity
ω	Radian frequency of turbulence variation
$\zeta(s)$	Riemann zeta function

Contents

Chapter 1

Introduction

Solving problems of wave propagation in turbulence is a field that occupies the services of a small group of researchers. The methods used in this community and the results obtained are not generally known by researchers in other communities. The main reason is that the field is considered difficult, and if there is not an obvious need to investigate the effects of turbulence, they are neglected. The difficulty arises from the need to solve stochastic differential equations. Advances made by Tatarski and Rytov reduce problems to multiple integrals. These integrals are often difficult to evaluate since fractional exponents of functions appear in integrands. The final step in most cases is to evaluate these integrals numerically and to present the results as parametric curves. Many cases are run to develop some insight into how a quantity of interest varies with parameters. Becoming an expert in this field requires a great deal of time to become familiar with these graphical results so that one has some insight into various effects.

As pointed out above, there is a formalism for reducing a problem to quadratures. This process is lengthy, and there are several ways of doing it. Different workers use different methods to get at the same result. This makes it difficult for the novice to understand the literature and to realize that there is some underlying order. This discourages a person with only a casual interest from developing a facility in this field. It was to make the solution of these problems more algorithmic that the methods expounded in this book were developed.

In this book I use the Rytov approximation to reduce a very general problem to a triple integral. I develop techniques that allow one to evaluate these integrals analytically.

The integrals that one encounters contain products of functions of which one or more is a Bessel function. Workers in the field look for these integrals in integral tables, and if unsuccessful, resort to numerical analysis. Even numerically, some of these integrals are difficult to evaluate. The integrand is often either the product of a function that goes to infinity multiplied by one that goes to zero at one of the integration limits, or the difference of two functions that each lead to a divergent integral. Great care must be exercised in evaluating these integrals.

The techniques developed in this book provide a recipe for obtaining analytical solutions. There is a saying, "You don't get something for nothing." Indeed, there is a price to be paid for being able to solve these problems more easily: The